# Precise Large Deviations of Aggregate Claims in a Size-Dependent Renewal Risk Model

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June 21, 2012

**Abstract**

Consider a renewal risk model in which claim sizes and inter-arrival times correspondingly form a sequence of independent and identically distributed random pairs, with each pair obeying a dependence structure described via the conditional distribution of the inter-arrival time given the subsequent claim size being large. We study large deviations of the aggregate amount of claims. For a heavy-tailed case, we obtain a precise large-deviation formula, which agrees with existing ones in the literature.

*Keywords:* Aggregate claims; Consistent variation; Dependence; Large deviations; Renewal counting process

*JEL classification*: C02

*Classification codes*: IM10, IM11

*Mathematics Subject Classifications*: Primary 60F10; Secondary 91B30, 60K05

## 1 Introduction

Consider the following renewal risk model. Claims of sizes $X_k$, $k \in \mathbb{N}$, arrive at successive renewal epochs with inter-arrival times $\theta_k$, $k \in \mathbb{N}$. Assume that $(X_k, \theta_k)$, $k \in \mathbb{N}$, form a sequence of independent and identically distributed (i.i.d.) copies of a generic random pair $(X, \theta)$ with marginal distribution functions $F = 1 - \overline{F}$ on $[0, \infty)$ and $G$ on $[0, \infty)$. Denote by $\tau_k = \sum_{i=1}^{k} \theta_i$, $k \in \mathbb{N}$, the claim arrival times, with $\tau_0 = 0$. Then the number of claims by time $t \geq 0$ is $N_t = \# \{k \in \mathbb{N} : \tau_k \leq t\}$, which forms an ordinary renewal counting process. In this way, the aggregate amount of claims is a random sum of the form

$$S_t = \sum_{k=1}^{N_t} X_k, \qquad t \geq 0, \tag{1.1}$$

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where, and throughout the paper, a summation over an empty index set produces a value 0 by convention.

Note that we do not assume independence between the claim size $X$ and the inter-arrival time $\theta$. Nevertheless, independence between the increments of the surplus process over claim arrival times is still preserved.

Such a non-standard renewal risk model was first proposed by Albrecher and Teugels (2006). It has recently attracted an increasing amount of attention from researchers in risk theory. See Boudreault et al. (2006), Cossette et al. (2008), Badescu et al. (2009), Asimit and Badescu (2010) and Li et al. (2010). In particular, Li et al. (2010) made the following assumption:

**Assumption 1.1** *There is a measurable function $h : [0, \infty) \longmapsto (0, \infty)$ such that the relation*

$$\Pr(X > x | \theta = t) \sim \Pr(X > x)\, h(t), \qquad x \to \infty, \tag{1.2}$$

*holds uniformly for $t \geq 0$.*

In relation (1.2), the symbol $\sim$ means that the quotient of both sides tends to 1 and the uniformity is understood as

$$\lim_{x\to\infty} \sup_{t\geq 0} \left| \frac{\Pr(X > x | \theta = t)}{\Pr(X > x)\, h(t)} - 1 \right| = 0.$$

As was pointed out by these authors, Assumption 1.1 defines a general dependence structure, which is easily verifiable for some commonly-used bivariate copulas and allows both positive and negative dependence. Since this structure is described via the conditional tail probability of a claim size given the inter-arrival time prior to the claim, the model is often termed as a time-dependent renewal risk model in the literature. By the way, integrating both sides of (1.2) with respect to $\Pr(\theta \in \mathrm{d}t)$ leads to $\mathrm{E}[h(\theta)] = 1$.

In the present paper, we make a different assumption on the dependence structure of $(X, \theta)$ as follows:

**Assumption 1.2** *There is a nonnegative random variable $\theta^*$ such that $\theta$ conditional on $(X > x)$ is stochastically bounded by $\theta^*$ for all large $x > 0$; in other words, there is some $x_0 > 0$ such that it holds for all $x > x_0$ and $t \in [0, \infty)$ that*

$$\Pr(\theta > t | X > x) \leq \Pr(\theta^* > t). \tag{1.3}$$

Assumption 1.2, in contrast to Assumption 1.1, describes a dependence structure via the conditional distribution of the inter-arrival time given the subsequent claim size being large. Hence, the model under Assumption 1.2 is termed by us as a size-dependent renewal risk model. Roughly speaking, Assumption 1.2 means that $X$ becoming large does not drag $\theta$ to infinity. A similar, but more restrictive, assumption has earlier appeared in Lemma 2.3 of Maulik et al. (2002).

Assumption 1.2 seems more natural than Assumption 1.1 in view of the perception that the waiting time for a large claim is dependent on the claim size but not vice versa. It is easy to see that Assumption 1.2 is more general than Assumption 1.1. Actually, let Assumption 1.1 be valid. As $x \to \infty$, it holds uniformly for all $t \in [0, \infty)$ that

$$\begin{aligned}\Pr(\theta > t | X > x) &= \frac{\Pr(X > x, \theta > t)}{\Pr(X > x)} \\ &= \int_t^\infty \frac{\Pr(X > x | \theta = s)}{\Pr(X > x)} \Pr(\theta \in \mathrm{d}s) \\ &\leq 2 \int_t^\infty h(s) \Pr(\theta \in \mathrm{d}s).\end{aligned}$$

Note that $G_0$ defined by $G_0(\mathrm{d}s) = h(s)\Pr(\theta \in \mathrm{d}s)$ is a proper distribution on $[0, \infty)$ since $\mathrm{E}[h(\theta)] = 1$ and that the right-hand side above is equal to $2\overline{G_0}(t)$. Then one can construct a nonnegative random variable $\theta^*$ distributed by $G^* = (1 - 2\overline{G_0}) \vee 0$ to serve as the stochastic upper bound for $\theta$ conditional on $(X > x)$ for all large $x$.

Recall that two random variables $X$ and $\theta$ distributed by $F$ and $G$, respectively, are called asymptotically independent (in the upper tail) if

$$\lim_{u \uparrow 1} \Pr\left(\theta > G^{\leftarrow}(u) | X > F^{\leftarrow}(u)\right) = 0,$$

where $F^{\leftarrow}(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\}$ with $\inf \emptyset = \infty$ by convention, and $G^{\leftarrow}(u)$ is understood similarly; see Section 5.2 of McNeil et al. (2005) for the definition. Clearly, Assumption 1.2 implies asymptotic independence of $(X, \theta)$, so does Assumption 1.1. We are not sure if we can weaken Assumption 1.2 in our Theorem 1.1 below to that $X$ and $\theta$ are asymptotically independent.

Various different non-standard renewal risk models have been proposed in the recent literature. The reader is referred to Asmussen et al. (1999), Chapter XIII of Asmussen and Albrecher (2010), Biard et al. (2008, 2011) and Asmussen and Biard (2011), among others.

We study large deviations of $\{S_t, t \geq 0\}$. We are only interested in the case of heavy-tailed claims. A useful heavy-tailed class is the class $\mathcal{C}$ of distribution functions with consistent variation (also called intermediate regular variation), characterized by the relations $\overline{F}(x) > 0$ for all $x \geq 0$ and

$$\lim_{y \uparrow 1} \limsup_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = \lim_{y \downarrow 1} \liminf_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

Note that the class $\mathcal{C}$ covers the well-known class $\mathcal{R}$ of distribution functions with regular variation.

Our main result is given below:

**Theorem 1.1** *Consider the aggregate amount of claims (1.1). In addition to Assumption 1.2, assume that* $F \in \mathcal{C}$, $\mathrm{E}[X] = \mu \in (0, \infty)$ *and* $\mathrm{E}[\theta] = 1/\lambda \in (0, \infty)$. *Then, for arbitrarily given* $\gamma > 0$, *it holds uniformly for all* $x \geq \gamma t$ *that*

$$\Pr(S_t - \mu\lambda t > x) \sim \lambda t \overline{F}(x), \qquad t \to \infty. \tag{1.4}$$

The study of precise large deviations of random sums was initiated by Klüppelberg and Mikosch (1997); see also Chapter 8.6 of Embrechts et al. (1997). Recent advances can be found in Tang et al. (2001), Ng et al. (2003, 2004), Kaas and Tang (2005), Liu (2007), Wang and Wang (2007), Lin (2008), Baltrūnas et al. (2008), Liu (2009) and Chen et al. (2011), among many others. A clear trend of the mainstream study is to incorporate various dependence structures among claims so as to more accurately reflect insurance practice.

To the best of our knowledge, the present work should be the first attempt to extend the study of precise large deviations to the case allowing (both positive and negative) dependence between claims and their inter-arrival times.

As was remarked by a few researchers in the area, precise large-deviation results of this type are particularly useful for evaluating some risk measures such as conditional tail expectation of the aggregate amount of claims from a large insurance portfolio. Finally, we would like to point out that our formula (1.4) agrees with existing ones in the literature. This indicates that the dependence structure of $(X, \theta)$ defined by Assumption 1.2 does not affect the asymptotic behavior of the large deviations of $\{S_t, t \geq 0\}$.

The rest of this paper consists of two sections. Section 2 recalls various preliminaries and prepares a few lemmas. Section 3 presents the proof of the main result by establishing corresponding asymptotic lower and upper bounds.

# 2 Preliminaries

Throughout this paper, for two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) \lesssim b(x)$ or $b(x) \gtrsim a(x)$ if $\limsup_{x\to\infty} a(x)/b(x) \leq 1$. As in Theorem 1.1, very often we equip a limit relation with certain uniformity, which is crucial for our purpose. For instance, for two positive bivariate functions $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$, we say that $a(x, t) \lesssim b(x, t)$, as $t \to \infty$, holds uniformly for $x \in \Delta_t \neq \varnothing$ if

$$\limsup_{t\to\infty} \sup_{x\in\Delta_t} \frac{a(x, t)}{b(x, t)} \leq 1.$$

Here we recall a useful functional index. Let $F$ be a distribution function on $(-\infty, \infty)$ and write $f = 1/\overline{F}$, which is a positive and non-decreasing function on $(-\infty, \infty)$. Let $J_F^+$ be the infimum of those $J$ for which there exists a constant $C = C(J)$ such that, for each $y_0 > 1$, the relation

$$\frac{f(xy)}{f(x)} \leq C(1 + o(1))y^J$$

holds uniformly in $y \in [1, y_0]$. The quantity $J_F^+$ defines the upper Matuszewska index of the function $f$. By Theorem 2.1.5 and Corollary 2.1.6 of Bingham et al. (1987), it coincides with

$$J_F^+ = \inf\left\{-\frac{\log \overline{F}_*(y)}{\log y} : y > 1\right\}, \qquad \text{where} \quad \overline{F}_*(y) = \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}.$$

Following Tang and Tsitsiashvili (2003), we simply call $J_F^+$ the upper Matuszewska index of $F$. Clearly, if $F \in \mathcal{C}$, then $J_F^+ < \infty$. From Proposition 2.2.1 of Bingham et al. (1987), we see that, for every $p > J_F^+$, there are positive constants $C$ and $x_0$ such that the inequality

$$\frac{\overline{F}(x)}{\overline{F}(xy)} \leq Cy^p \tag{2.1}$$

holds for all $xy \geq x \geq x_0$. Then one can easily see that the relation

$$x^{-p} = o(\overline{F}(x)), \qquad x \to \infty, \tag{2.2}$$

holds for all $p > J_F^+$. See also Lemma 3.5 of Tang and Tsitsiashvili (2003) for more details.

Recall Assumption 1.2. Introduce a nonnegative random variable $\theta_1^*$ that is independent of all sources of randomness and is identically distributed as $\theta$ conditional on $(X > x)$. Correspondingly, write $\tau_1^* = \theta_1^*$, $\tau_k^* = \theta_1^* + \sum_{i=2}^k \theta_i$ for $k = 2, 3, \ldots$ and define

$$N_t^* = \#\left\{k \in \mathbb{N} : \tau_k^* \leq t\right\}, \qquad t \geq 0.$$

Note that $\{N_t^*, t \geq 0\}$ is a so-called delayed renewal counting process in that its first inter-arrival time is not necessarily identically distributed as, though still independent of, the other inter-arrival times. Note also that the distribution of $\theta_1^*$ depends on $x$ through the condition $(X > x)$, so does the distribution of $\{N_t^*, t \geq 0\}$.

The following lemma establishes the law of large numbers for $\{N_t^*, t \geq 0\}$:

**Lemma 2.1** *In addition to Assumption 1.2, assume that* $\mathrm{E}[\theta] = 1/\lambda \in (0, \infty)$. *Then it holds for every* $0 < \delta < \lambda$ *and every function* $\gamma(\cdot) : [0, \infty) \longmapsto (0, \infty)$ *with* $\gamma(t) \uparrow \infty$ *as* $t \to \infty$ *that*

$$\lim_{t\to\infty} \sup_{x\geq\gamma(t)} \Pr\left(\left|\frac{N_t^*}{t} - \lambda\right| > \delta\right) = 0. \tag{2.3}$$

**Proof.** For a real number $y$, denote by $\lfloor y \rfloor$ its integer part. Observe that, for all large $t$,

$$\begin{aligned}\Pr\left(\left|\frac{N_t^*}{t}-\lambda\right|>\delta\right) &= \Pr\left(N_t^*<(\lambda-\delta)t\right)+\Pr\left(N_t^*>(\lambda+\delta)t\right)\\ &\le \Pr\left(\theta_1^*+\sum_{i=2}^{\lfloor(\lambda-\delta)t\rfloor+1}\theta_i>t\right)+\Pr\left(\theta_1^*+\sum_{i=2}^{\lfloor(\lambda+\delta)t\rfloor}\theta_i\le t\right)\\ &\le \Pr\left(\theta^*+\sum_{i=2}^{\lfloor(\lambda-\delta)t\rfloor+1}\theta_i>t\right)+\Pr\left(\sum_{i=2}^{\lfloor(\lambda+\delta)t\rfloor}\theta_i\le t\right),\end{aligned}$$

where in the last step we used an independent and nonnegative random variable $\theta^*$ justified by Assumption 1.2 to bound $\theta_1^*$. By the law of large numbers for the partial sums $\sum_{i=1}^n \theta_i$, $n\in\mathbb{N}$, both probabilities on the right-hand side above converge to zero. Thus, relation (2.3) holds. ■

The following lemma, which is a restatement of Theorem 3.1 of Ng et al. (2004), forms the basis for the proof of Theorem 1.1:

**Lemma 2.2** *Let $\{X_k, k\in\mathbb{N}\}$ be a sequence of i.i.d. nonnegative random variables with common distribution function $F\in\mathcal{C}$ and mean $\mu\in(0,\infty)$. Then, for arbitrarily given $\gamma>0$, it holds uniformly for $x\ge\gamma n$ that*

$$\Pr\left(S_n-n\mu>x\right)\sim n\overline{F}(x),\qquad n\to\infty.$$

The following lemma addresses a similar problem as Lemma 4.4 of Li et al. (2010) but, restricted to the case with $J_F^+<\infty$, it shows a much sharper upper bound than the latter:

**Lemma 2.3** *Let $(X_k,\theta_k)$, $k\in\mathbb{N}$, be i.i.d. copies of a generic random pair $(X,\theta)$, where $X$ is distributed by $F$ with upper Matuszewska index $J_F^+<\infty$ and $\theta$ is nonnegative. Then, for every $p>J_F^+$, there is some constant $C>0$ such that, uniformly for all $x\ge0$, $t\ge0$ and $n\in\mathbb{N}$,*

$$\Pr\left(\sum_{k=1}^n X_k>x,\tau_n\le t\right)\le Cn^{p+1}\overline{F}(x)\Pr\left(\tau_{n-1}\le t\right).$$

**Proof.** By the i.i.d. assumption on the sequence $\{(X_k,\theta_k),k\in\mathbb{N}\}$ and the nonnegativity of $\theta$, we have

$$\begin{aligned}\Pr\left(\sum_{k=1}^n X_k>x,\tau_n\le t\right) &\le \sum_{k=1}^n\Pr\left(X_k>\frac{x}{n},\tau_n\le t\right)\\ &\le \sum_{k=1}^n\Pr\left(X_k>\frac{x}{n},\sum_{i=1,i\neq k}^n\theta_i\le t\right)\\ &= n\Pr\left(X>\frac{x}{n}\right)\Pr\left(\tau_{n-1}\le t\right).\end{aligned}\tag{2.4}$$

By inequality (2.1), for arbitrarily fixed $p > J_F^+$, there are some large positive constants $C_1$ and $x_0$ such that the inequality $\Pr(X > x/n) \leq C_1 n^p \overline{F}(x)$ holds for all $x \geq n x_0$. Hence,

$$\begin{aligned}\Pr\left(X > \frac{x}{n}\right) &\leq 1_{(x<nx_0)} + \Pr\left(X > \frac{x}{n}\right) 1_{(x \geq n x_0)} \\ &\leq \frac{(nx_0)^p}{x^p} + C_1 n^p \overline{F}(x) \\ &\leq C n^p \overline{F}(x)\end{aligned}$$

for some $C > C_1$, where the last step is due to relation (2.2). Substituting this into (2.4) yields the desired inequality. ■

The last lemma below is a restatement of Theorem 1(i) of Kočetova et al. (2009):

**Lemma 2.4** *Let the inter-arrival times $\theta_k$, $k \in \mathbb{N}$, form a sequence of i.i.d. nonnegative random variables with common mean $1/\lambda \in (0, \infty)$. Then, it holds for every $a > \lambda$ and some $b > 1$ that*

$$\lim_{t\to\infty} \sum_{n>at} b^n \Pr(\tau_n \leq t) = 0.$$

# 3 Proof of Theorem 1.1

Throughout this section, unless otherwise stated, every limit relation is understood as valid uniformly for all $x \geq \gamma t$ as $t \to \infty$. Trivially, relation (1.4) amounts to the conjunction of

$$\Pr(S_t - \mu\lambda t > x) \gtrsim \lambda t \overline{F}(x) \qquad \text{and} \qquad \Pr(S_t - \mu\lambda t > x) \lesssim \lambda t \overline{F}(x), \tag{3.1}$$

which will be proven separately in the following two subsections.

## 3.1 Proof of the first relation in (3.1)

Let $0 < \delta < 1$ and $v > 1$ be arbitrarily fixed with $\delta$ small. We derive

$$\begin{aligned}\Pr(S_t - \mu\lambda t > x) &\geq \sum_{(1-\delta)\lambda t \leq n \leq (1+\delta)\lambda t} \Pr\left(\sum_{k=1}^n X_k - \mu\lambda t > x, N_t = n\right) \\ &\geq \sum_{(1-\delta)\lambda t \leq n \leq (1+\delta)\lambda t} \Pr\left(\sum_{k=1}^n X_k - \mu\lambda t > x, N_t = n, \bigvee_{i=1}^n X_i > vx\right).\end{aligned}$$

Hence, by Bonferroni's inequality,

$$\Pr(S_t - \mu\lambda t > x) \geq \sum_{(1-\delta)\lambda t \leq n \leq (1+\delta)\lambda t} (I_1(x,t;n) - I_2(x,t;n)) \tag{3.2}$$

with

$$I_1(x,t;n) = \sum_{i=1}^{n} \Pr\left(\sum_{k=1}^{n} X_k - \mu\lambda t > x, N_t = n, X_i > vx\right),$$
$$I_2(x,t;n) = \sum_{1\le i<j\le n} \Pr\left(N_t = n, X_i > vx, X_j > vx\right).$$

First deal with the sum of $I_1(x,t;n)$ in (3.2). Similarly as in Lemma 2.1, introduce a nonnegative random variable $\theta_1^*$ that is independent of all sources of randomness and is identically distributed as $\theta$ conditional on $(X > vx)$, and then define a delayed renewal counting process $\{N_t^*, t \ge 0\}$, which differs from $\{N_t, t \ge 0\}$ only by the first inter-arrival time $\theta_1^*$. By conditioning on $(X_i > vx)$ and applying the i.i.d. assumption of the sequence $\{(X_k, \theta_k), k \in \mathbb{N}\}$, we have, for all large $t$,

$$\begin{aligned}
&I_1(x,t;n)\\
&\ge \sum_{i=1}^{n} \Pr\left(\sum_{k=1,k\ne i}^{n} X_k - \mu\lambda t > (1-v)x, N_t = n, X_i > vx\right)\\
&= \sum_{i=1}^{n} \Pr\left(\sum_{k=1,k\ne i}^{n} X_k - \mu\lambda t > (1-v)x, N_t = n \,\middle|\, X_i > vx\right) \Pr\left(X_i > vx\right)\\
&= n\overline{F}(vx) \Pr\left(\sum_{k=2}^{n} X_k - \mu\lambda t > (1-v)x, N_t^* = n\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{(1-\delta)\lambda t\le n\le(1+\delta)\lambda t} I_1(x,t;n)\\
&\gtrsim \sum_{(1-\delta)\lambda t\le n\le(1+\delta)\lambda t} n\overline{F}(vx) \Pr\left(\sum_{k=2}^{n} X_k - \mu\lambda t > (1-v)x, N_t^* = n\right)\\
&\ge (1-\delta)\lambda t\overline{F}(vx) \Pr\left(\sum_{2\le k\le(1-\delta)\lambda t} X_k - \mu\lambda t > (1-v)x,\ \left|\frac{N_t^*}{\lambda t} - 1\right| \le \delta\right)\\
&\ge (1-\delta)\lambda t\overline{F}(vx) \left(\Pr\left(\sum_{2\le k\le(1-\delta)\lambda t} X_k - \mu\lambda t > (1-v)x\right) - \Pr\left(\left|\frac{N_t^*}{\lambda t} - 1\right| > \delta\right)\right),
\end{aligned}$$

where in the last step we applied the elementary inequality $\Pr(A \cap B) \ge \Pr(A) - \Pr(B^c)$ for two arbitrary events $A$ and $B$. For some positive $\delta$ small enough such that $(1-\delta)\mu\lambda - \mu\lambda > (1-v)\gamma$, by the law of large numbers for the partial sums $\sum_{k=1}^{n} X_k$, $n \in \mathbb{N}$, the first probability on the right-hand side above is not less than

$$\Pr\left(\frac{1}{t}\sum_{2\le k\le(1-\delta)\lambda t} X_k - \mu\lambda > (1-v)\gamma\right) \to 1.$$

By Lemma 2.1, the second probability tends to 0. It follows that

$$\sum_{(1-\delta)\lambda t\leq n\leq(1+\delta)\lambda t} I_1(x,t;n) \gtrsim (1-\delta)\lambda t\overline{F}(vx). \tag{3.3}$$

We turn to the sum of $I_2(x,t;n)$ in (3.2). Interchanging the order of summations yields

$$\begin{aligned}
&\sum_{(1-\delta)\lambda t\leq n\leq(1+\delta)\lambda t} I_2(x,t;n)\\
&= \left(\overline{F}(vx)\right)^2 \sum_{(1-\delta)\lambda t\leq n\leq(1+\delta)\lambda t}\ \sum_{1\leq i<j\leq n} \Pr\left(N_t=n|\, X_i>vx, X_j>vx\right)\\
&\leq \left(\overline{F}(vx)\right)^2 \sum_{1\leq i<j\leq(1+\delta)\lambda t}\ \sum_{(1-\delta)\lambda t\leq n\leq(1+\delta)\lambda t} \Pr\left(N_t=n|\, X_i>vx, X_j>vx\right)\\
&\leq \left(\overline{F}(vx)\right)^2 \left((1+\delta)\lambda t\right)^2\\
&= o(1)t\overline{F}(vx),
\end{aligned} \tag{3.4}$$

where the last step is due to the fact that $t\overline{F}(vx)\leq \gamma^{-1}x\overline{F}(vx)\to 0$ since $\mu<\infty$. Finally, substituting (3.3) and (3.4) into (3.2) yields

$$\Pr\left(S_t-\mu\lambda t>x\right) \gtrsim (1-\delta)\lambda t\overline{F}(vx) - o(1)t\overline{F}(vx).$$

By the arbitrariness of $\delta$ and $v$ and the condition $F\in\mathcal{C}$, we obtain the first relation in (3.1).

## 3.2 Proof of the second relation in (3.1)

For arbitrarily fixed, but small, $0<\delta<1$, we split $\Pr\left(S_t-\mu\lambda t>x\right)$ into two parts as

$$\begin{aligned}
&\Pr\left(S_t-\mu\lambda t>x\right)\\
&= \Pr\left(\sum_{k=1}^{N_t} X_k-\mu\lambda t>x, N_t\leq(1+\delta)\lambda t\right) + \Pr\left(\sum_{k=1}^{N_t} X_k-\mu\lambda t>x, N_t>(1+\delta)\lambda t\right)\\
&= J_1(x,t)+J_2(x,t).
\end{aligned} \tag{3.5}$$

As in the proof of Lemma 2.1, denote by $\lfloor y\rfloor$ the integer part of a real number $y$. For some small $\delta$ such that $\gamma-\delta\mu\lambda>0$, by Lemma 2.2 we have

$$\begin{aligned}
J_1(x,t) &\leq \Pr\left(\sum_{1\leq k\leq(1+\delta)\lambda t} X_k-\mu\lambda t>x\right)\\
&= \Pr\left(\sum_{k=1}^{\lfloor(1+\delta)\lambda t\rfloor} X_k-\mu\lfloor(1+\delta)\lambda t\rfloor > x+\mu\lambda t-\mu\lfloor(1+\delta)\lambda t\rfloor\right)\\
&\sim \lfloor(1+\delta)\lambda t\rfloor\,\overline{F}(x+\mu\lambda t-\mu\lfloor(1+\delta)\lambda t\rfloor)\\
&\leq (1+\delta)\lambda t\overline{F}\left(x\left(1-\delta\mu\lambda/\gamma\right)\right).
\end{aligned} \tag{3.6}$$

By Lemmas 2.3 and 2.4, it holds for every $p > J_F^+$ that

$$\begin{aligned}
J_2(x,t) &= \sum_{n>(1+\delta)\lambda t} \Pr\left(\sum_{k=1}^{n} X_k - \mu\lambda t > x, N_t = n\right) \\
&\le \sum_{n>(1+\delta)\lambda t} \Pr\left(\sum_{k=1}^{n} X_k > x, \tau_n \le t\right) \\
&\le C\overline{F}(x) \sum_{n>(1+\delta)\lambda t} n^{p+1} \Pr\left(\tau_{n-1} \le t\right) \\
&= o\left(\overline{F}(x)\right) \\
&= o\left(\lambda t \overline{F}(x)\right). \qquad (3.7)
\end{aligned}$$

Substituting (3.6) and (3.7) into (3.5) yields

$$\Pr\left(S_t - \mu\lambda t > x\right) \lesssim (1+\delta)\lambda t \overline{F}\left(x\left(1 - \delta\mu\lambda/\gamma\right)\right) + o\left(\lambda t \overline{F}(x)\right).$$

By the arbitrariness of $\delta$ and the condition $F \in \mathcal{C}$, we obtain the second relation in (3.1).

**Acknowledgments.** The authors are indebted to a referee for his/her very useful suggestions on earlier versions of this paper. In particular, as suggested by the referee, we have conducted the research under Assumption 1.2, which is more general than Assumption 1.1, and we have significantly shortened the proof of Lemma 2.3.

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