Precise Large Deviations of Random Sums in Presence of Negative Dependence and Consistent Variation

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Abstract

The study of precise large deviations for random sums is an important topic in insurance and finance. In this paper, we extend recent results of Tang (2006) and Liu (2009) to random sums in various situations. In particular, we establish a precise large deviation result for a nonstandard renewal risk model in which innovations, modelled as real-valued random variables, are negatively dependent with common consistently-varying-tailed distribution, and their inter-arrival times are also negatively dependent.

Keywords: Consistent variation; counting process; extended lower/upper negative dependence; precise large deviation; uniformity.

1 Introduction

We say that a distribution F on $(-\infty,\infty)$ has a consistently-varying tail, written as $F\in\mathcal{C}$, if $\overline{F}(x)=1-F(x)>0$ for all x and

$$\lim_{y\searrow 1}\liminf_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)}=1,\quad \text{or, equivalently,}\quad \lim_{y\nearrow 1}\limsup_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)}=1.$$

Clearly, the class $\mathcal{C}$ covers the famous class $\mathcal{R}$ of distributions with regularly-varying tails in the sense that the relation

$$\lim_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)}=y^{-\alpha}$$

holds for some $\alpha\geq 0$ and all $y>0$.

According to Ebrahimi and Ghosh (1981) and Block et al. (1982), random variables $\{X_k,k=1,2,\ldots\}$ are said to be lower negatively dependent (LND) if for each $n=1,2,\ldots$

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and all $x_1, \ldots, x_n$,

$$\Pr\left(\bigcap_{k=1}^{n}(X_k \le x_k)\right) \le \prod_{k=1}^{n}\Pr(X_k \le x_k);$$

they are said to be upper negatively dependent (UND) if for each $n = 1, 2, \ldots$ and all $x_1, \ldots, x_n$,

$$\Pr\left(\bigcap_{k=1}^{n}(X_k > x_k)\right) \le \prod_{k=1}^{n}\Pr(X_k > x_k);$$

and they are said to be negatively dependent (ND) if they are both LND and UND. Note that for $n = 2$, the LND, UND, and ND structures are equivalent; see, for example, Lehmann (1966).

Throughout this paper, for a sequence of random variables $\{X_k, k = 1, 2, \ldots\}$, denote by $S_n = X_1 + \cdots + X_n$ its nth partial sum. Tang (2006) obtained the following result. Let $\{X_k, k = 1, 2, \ldots\}$ be a sequence of ND random variables with common distribution $F \in \mathcal{C}$ having zero mean and satisfying

$$xF(-x) = o\left(\overline{F}(x)\right), \qquad x \to \infty.$$

Then, for each fixed $\gamma > 0$, the relation

$$\Pr(S_n > x) \sim n\overline{F}(x), \qquad n \to \infty, \tag{1.1}$$

holds uniformly for all $x \ge \gamma n$; that is,

$$\lim_{n\to\infty}\sup_{x\ge\gamma n}\left|\frac{\Pr(S_n > x)}{n\overline{F}(x)} - 1\right| = 0.$$

Recently, Liu (2009) extended this result to a slightly more general dependence structure. In her paper, random variables $\{X_k, k = 1, 2, \ldots\}$ are said to be extended lower negatively dependent (ELND) if there is some $M > 0$ such that, for each $n = 1, 2, \ldots$ and all $x_1, \ldots, x_n$,

$$\Pr\left(\bigcap_{k=1}^{n}(X_k \le x_k)\right) \le M\prod_{k=1}^{n}\Pr(X_k \le x_k);$$

they are said to be extended upper negatively dependent (EUND) if there is some $M > 0$ such that, for each $n = 1, 2, \ldots$ and all $x_1, \ldots, x_n$,

$$\Pr\left(\bigcap_{k=1}^{n}(X_k > x_k)\right) \le M\prod_{k=1}^{n}\Pr(X_k > x_k);$$

and they are said to be extended negatively dependent (END) if they are both ELND and EUND. Furthermore, Liu (2009) considered the case with non-identically distributed random variables, but in doing so the author had to impose an extra condition showing that the underlying distributions do not differ too much from each other.

For simplicity, we restrict our attention to the case with identically distributed random variables in this paper. In this case, Theorem 2.1 of Liu (2009) can be restated as follows. Let $\{X_k, k = 1, 2, \ldots\}$ be a sequence of END random variables with common distribution $F \in \mathcal{C}$ having zero mean and satisfying

$$F(-x) = o\left(\overline{F}(x)\right) \text{ as } x \to \infty, \quad \mathrm{E}\,|X_1|^r\, 1_{(X_1 \le 0)} < \infty \text{ for some } r > 1. \tag{1.2}$$

Then, for every fixed $\gamma > 0$, relation (1.1) holds uniformly for all $x \ge \gamma n$.

We aim at extending the above-mentioned results of Tang (2006) and Liu (2009) to random sums

$$S_{N_t} = \sum_{k=1}^{N_t} X_k, \qquad t \ge 0,$$

where $\{N_t, t \ge 0\}$, independent of $\{X_k, k = 1, 2, \ldots\}$, is a counting process (that is, a non-negative, non-decreasing, and integer-valued stochastic process) with a finite mean function λ_t for $t \ge 0$ but $\lambda_t \to \infty$ as $t \to \infty$. We shall establish precise large deviation results for S_{N_t}.

The study of precise large deviations for random sums was initiated by Klüppelberg and Mikosch (1997), who presented several applications in insurance and finance. See also Embrechts et al. (1997, Chapter 8) and Mikosch and Nagaev (1998). We remark that results of precise large deviations for random sums are particularly useful for evaluation of some risk measures such as conditional tail expectation and value at risk of aggregate claims of a large insurance portfolio. See McNeil et al. (2005) for a review of risk measures. Recent advances on precise large deviations for random sums can be found in Tang et al. (2001), Ng et al. (2003, 2004), Liu and Hu (2003), Liu (2007), Wang and Wang (2007), Lin (2008), Baltrūnas et al. (2008), Shen and Lin (2008), among others.

The rest of this paper consists of four sections. Section 2 gives some preliminaries. Sections 3-4 derive a few results of precise large deviations for random sums in our set-up. Section 5 shows an important special case with a quasi-renewal model.

2 Preliminaries

Throughout this paper, for two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(\cdot) \sim b(\cdot)$ if $\lim a(\cdot)/b(\cdot) = 1$, write $a(\cdot) \lesssim b(\cdot)$ or $b(\cdot) \gtrsim a(\cdot)$ if $\limsup a(\cdot)/b(\cdot) \le 1$, and write $a(\cdot) \asymp b(\cdot)$ if $0 < \liminf a(\cdot)/b(\cdot) \le \limsup a(\cdot)/b(\cdot) < \infty$. Frequently, to enhance the theoretical or applied interest of the asymptotic relations under consideration, we equip them with certain uniformity, which is crucial for our purpose. For instance, for two positive bivariate functions

$a(t,x)$ and $b(t,x)$, we say $a(t,x) \sim b(t,x)$ holds as $t \to \infty$ uniformly for all $x \in D_t \neq \varnothing$ if

$$\lim_{t\to\infty} \sup_{x\in D_t} \left| \frac{a(t,x)}{b(t,x)} - 1 \right| = 0.$$

For a distribution F, we follow the style of Tang and Tsitsiashvili (2003) to define

$$J_F^+ = -\lim_{v\to\infty} \log \frac{\overline{F}_*(v)}{\log v} \qquad \text{with} \quad \overline{F}_*(v) = \liminf_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} \qquad \text{for } v > 0,$$

and call the quantity J_F^+ the upper Matuszewska index of the distribution F. For details of the Matuszewska indices, see Bingham et al. (1987, Chapter 2.1). For further discussions and applications, see Cline and Samorodnitsky (1994) and Tang and Tsitsiashvili (2003).

Clearly, if $F \in \mathcal{C}$ then $J_F^+ < \infty$. Moreover, by Lemma 3.5 of Tang and Tsitsiashvili (2003), it holds for every $p > J_F^+$ that

$$x^{-p} = o\left(\overline{F}(x)\right), \qquad x \to \infty, \tag{2.1}$$

and $J_F^+ \geq 1$ if the distribution $F(x) = F(x)1_{(x\geq 0)}$ has a finite mean.

Lemma 2.1. *Let $\{\xi_t, t \geq 0\}$ be a nonnegative stochastic process with $\mathrm{E}\xi_t \to 1$ as $t \to \infty$. Then, the following statements are equivalent:*

(i) $\xi_t \xrightarrow{P} 1$ as $t \to \infty$;

(ii) $\mathrm{E}\xi_t 1_{(\xi_t > 1+\varepsilon)} = o(1)$ as $t \to \infty$ for every fixed $\varepsilon > 0$; and

(iii) $\Pr(\xi_t \leq 1 - \delta) = o(1)$ as $t \to \infty$ for every fixed $\delta \in (0,1)$.

Proof. The equivalence of (i) and (ii) can be found in Lemma 3.1 of Ng et al. (2003). For self-containedness, we give a complete proof of the present lemma by showing the following order of implications: (i)⇒(ii)⇒(iii)⇒(i).

(i)⇒(ii): For every fixed $\varepsilon > 0$, an application of the dominated convergence theorem gives

$$\mathrm{E}\xi_t 1_{(\xi_t>1+\varepsilon)} = \mathrm{E}\xi_t - \mathrm{E}\xi_t 1_{(\xi_t\leq 1+\varepsilon)} \to 0, \qquad t \to \infty.$$

(ii)⇒(iii): For every fixed $\varepsilon > 0$ and $\delta \in (0,1)$, we have

$$\begin{aligned} \mathrm{E}\xi_t &= \mathrm{E}\xi_t 1_{(\xi_t\leq 1-\delta)} + \mathrm{E}\xi_t 1_{(1-\delta<\xi_t\leq 1+\varepsilon)} + \mathrm{E}\xi_t 1_{(\xi_t>1+\varepsilon)} \\ &\leq (1-\delta)\Pr(\xi_t \leq 1-\delta) + (1+\varepsilon)(1 - \Pr(\xi_t \leq 1-\delta)) + \mathrm{E}\xi_t 1_{(\xi_t>1+\varepsilon)}. \end{aligned} \tag{2.2}$$

Then, it follows from (ii) and $\mathrm{E}\xi_t \to 1$ as $t \to \infty$ that

$$\limsup_{t\to\infty} \Pr(\xi_t \leq 1-\delta) \leq \frac{\varepsilon}{\varepsilon+\delta}.$$

Letting $\varepsilon \downarrow 0$ yields (iii).

(iii)⇒(i): It suffices to show that, for every fixed $\varepsilon > 0$,

$$\lim_{t\to\infty} \Pr\left(\xi_t > 1+\varepsilon\right) = 0. \tag{2.3}$$

Similar to the derivation of (2.2), for every fixed $\varepsilon > 0$ and $\delta \in (0,1)$, we have

$$\begin{aligned} \mathrm{E}\xi_t &\geq \mathrm{E}\xi_t 1_{(1-\delta<\xi_t\leq 1+\varepsilon)} + \mathrm{E}\xi_t 1_{(\xi_t>1+\varepsilon)} \\ &\geq (1-\delta)\left(1-\Pr\left(\xi_t \leq 1-\delta\right) - \Pr\left(\xi_t > 1+\varepsilon\right)\right) + (1+\varepsilon)\Pr\left(\xi_t > 1+\varepsilon\right). \end{aligned}$$

Since $\mathrm{E}\xi_t \to 1$ as $t \to \infty$, it follows from (iii) that

$$\limsup_{t\to\infty} \Pr\left(\xi_t > 1+\varepsilon\right) \leq \frac{\delta}{\varepsilon+\delta}.$$

Finally, letting $\delta \downarrow 0$ leads to (2.3). □

Lemma 2.2. *Let $\{X_k, k = 1,2,\ldots\}$ be a sequence of EUND random variables with common distribution F and zero mean. If $\mathrm{E}X_1^r 1_{(X_1\geq 0)} < \infty$ for some $r > 1$, then, for every fixed $\gamma > 0$ and $p > 0$, there exist positive numbers v and $C = C(v,\gamma)$ such that, for all $n = 1,2,\ldots$ and $x \geq \gamma n$,*

$$\Pr\left(S_n > x\right) \leq n\overline{F}\left(vx\right) + Cx^{-p}.$$

To prove the lemma, one can use arguments similar to those in the proof of Lemma 2.3 of Tang (2006) with some modifications in relation to Lemma 3.1 of Liu (2009). Therefore, we omit the proof here.

Lemma 2.3. *Let $\{X_k, k = 1,2,\ldots\}$ be a sequence of EUND random variables with common distribution F. If $0 < m_+ = \mathrm{E}X_1 1_{(X_1\geq 0)} < \infty$, then, for every fixed $v > 0$ and some $C = C(v) > 0$, the inequality*

$$\Pr\left(S_n > x\right) \leq n\overline{F}\left(vx\right) + C\left(n/x\right)^{1/v} \tag{2.4}$$

holds for all $n = 1,2,\ldots$ and $x > 0$.

Proof. Write $\widetilde{X_k} = X_k 1_{(0<X_k\leq vx)} + vx 1_{(X_k>vx)}$ for $k = 1,2,\ldots$, which are still EUND, and write $\widetilde{S_n} = \sum_{k=1}^n \widetilde{X_k}$. Hence, employing a standard truncation argument and Lemma 3.1 of Liu (2009), we obtain, respectively,

$$\Pr\left(S_n > x\right) \leq n\overline{F}\left(vx\right) + \Pr\left(\widetilde{S_n} > x\right), \tag{2.5}$$

and

$$\Pr\left(\widetilde{S_n} > x\right) \leq M\mathrm{e}^{-hx}\left(\mathrm{E}\mathrm{e}^{h\widetilde{X_1}}\right)^n, \tag{2.6}$$

for some $M > 0$ and $h = h(x, v) > 0$. By the monotonicity in $y \in (0, \infty)$ of $(\mathrm{e}^{hy} - 1)/y$,

$$\begin{aligned}
\mathrm{E}\mathrm{e}^{h\widetilde{X_1}} &= \int_0^{vx} \frac{\mathrm{e}^{hy} - 1}{y} yF(\mathrm{d}y) + \left(\mathrm{e}^{hvx} - 1\right) \overline{F}(vx) + 1 \\
&\leq \frac{\mathrm{e}^{hvx} - 1}{vx} m_+ + \left(\mathrm{e}^{hvx} - 1\right) \overline{F}(vx) + 1. \qquad (2.7)
\end{aligned}$$

Substituting (2.7) into (2.6) yields

$$\Pr\left(\widetilde{S_n} > x\right) \leq M \exp\left\{\frac{\mathrm{e}^{hvx} - 1}{vx} nm_+ + \left(\mathrm{e}^{hvx} - 1\right) n\overline{F}(vx) - hx\right\}.$$

Then, setting $h = (vx)^{-1} \log(x(nm_+)^{-1} + 1) > 0$ gives

$$\Pr\left(\widetilde{S_n} > x\right) \leq M \exp\left\{\frac{1}{v} + \frac{x}{m_+}\overline{F}(vx) - \frac{1}{v}\log\left(\frac{x}{nm_+} + 1\right)\right\} \leq C\left(\frac{n}{x}\right)^{1/v}, \qquad (2.8)$$

with the coefficient C given by

$$C = \sup_{x \geq 0} M \exp\left\{\frac{1}{v} + \frac{x}{m_+}\overline{F}(vx) + \frac{1}{v}\log m_+\right\} < \infty.$$

Substituting (2.8) into (2.5) yields (2.4). □

3 Random sums of nonnegative random variables

Let $\{Z_k, k = 1, 2, \ldots\}$ be a sequence of nonnegative and END random variables with common distribution $F \in \mathcal{C}$ and finite mean $\mu > 0$. If we put $Z_k - \mu = X_k$, then the two relations in (1.2) hold automatically. By (1.1), for every fixed $\gamma > 0$, the relation

$$\Pr\left(\sum_{k=1}^{n} Z_k - n\mu > x\right) \sim n\overline{F}(x), \qquad n \to \infty, \qquad (3.1)$$

holds uniformly for all $x \geq \gamma n$. The following theorem extends (3.1) to random sums.

Theorem 3.1. *Let $\{Z_k, k = 1, 2, \ldots\}$ be a sequence of nonnegative and END random variables with common distribution $F \in \mathcal{C}$ and finite mean $\mu > 0$, and let $\{N_t, t \geq 0\}$ be a counting process independent of $\{Z_k, k = 1, 2, \ldots\}$ and satisfying*

$$\mathrm{E}N_t^p 1_{(N_t > (1+\delta)\lambda_t)} = O(\lambda_t), \qquad t \to \infty, \qquad (3.2)$$

for some $p > J_F^+$ and all $\delta > 0$. Then, for every fixed $\gamma > 0$, the relation

$$\Pr\left(\sum_{k=1}^{N_t} Z_k - \mu\lambda_t > x\right) \sim \lambda_t\overline{F}(x), \qquad t \to \infty,$$

holds uniformly for all $x \geq \gamma\lambda_t$.

To prove Theorem 3.1, one can mimic the proof of Theorem 4.1 of Ng et al. (2004). Therefore, we omit the proof here.

Condition (3.2) first appeared in Tang et al. (2001) for weakening corresponding conditions in Klüppelberg and Mikosch (1997). This condition is fulfilled at least by the commonly-used renewal counting process; see Lemma 3.5 of Tang et al. (2001). By Lemma 2.1, condition (3.2) implies the weak law of large numbers,

$$\frac{N_t}{\lambda_t} \xrightarrow{P} 1, \qquad t \to \infty. \tag{3.3}$$

Furthermore, by Lemma 4.1 of Kaas and Tang (2005), condition (3.2) is equivalent to

$$\mathrm{E}\left(N_t - (1+\delta)\lambda_t\right)^p 1_{(N_t > (1+\delta)\lambda_t)} = O\left(\lambda_t\right), \qquad t \to \infty, \tag{3.4}$$

for some $p > J_F^+$ and all $\delta > 0$. In view of the fact that the left-hand side of (3.4) is a convex function of N_t, this enables us to apply some well-known results in stochastic ordering theory to verify the condition.

By (1.1), with an arbitrarily fixed constant $b > 0$, the relation

$$\Pr\left(\sum_{k=1}^{n} (Z_k - \mu - b) > x\right) \sim n\overline{F}(x + bn), \qquad n \to \infty, \tag{3.5}$$

holds uniformly for all $x \geq 0$. The constant b in (3.5) may be interpreted as the safety loading if we treat each Z_k as the amount of aggregate claims coming from the kth insurance policy. Related discussions but for the i.i.d. case can be found in Ng et al. (2003). In Parallel with Theorem 3.1, we have the following result which extends (3.5) to random sums.

Theorem 3.2. *Let $\{Z_k, k = 1, 2, \ldots\}$ be a sequence of nonnegative and END random variables with common distribution $F \in \mathcal{C}$ and finite mean $\mu > 0$, and let $\{N_t, t \geq 0\}$ be a counting process independent of $\{Z_k, k = 1, 2, \ldots\}$ and satisfying (3.3), the weak law of large numbers. Then, for every fixed $\gamma > 0$ and $b > 0$, the relation*

$$\Pr\left(\sum_{k=1}^{N_t} (Z_k - \mu - b) > x\right) \sim \lambda_t\overline{F}(x + b\lambda_t), \qquad t \to \infty, \tag{3.6}$$

holds uniformly for all $x \geq \gamma\lambda_t$.

The proof of Theorem 3.2 is similar to that of Theorem 2.2 of Ng et al. (2003). Therefore, we omit the proof here.

4 Random sums of real-valued random variables

In the random sum appearing in (3.6), each term can be thought of as a random variable $X_k = Z_k - \mu$ with zero mean centered by a constant b. Motivated by Theorem 3.2, in this section we establish a general result for random sums of END random variables with zero mean centered by a constant.

Theorem 4.1. *Let* $\{X_k, k = 1, 2, \ldots\}$ *be a sequence of END random variables with common distribution* $F \in \mathcal{C}$ *having zero mean and satisfying (1.2). Also, let* $\{N_t, t \geq 0\}$ *be a counting process independent of* $\{X_k, k = 1, 2, \ldots\}$*. For a real number* c*, consider the random sum* $S_{N_t,c} = \sum_{k=1}^{N_t} (X_k + c)$*,* $t \geq 0$*. The relation*

$$\Pr(S_{N_t,c} > x) \sim \lambda_t \overline{F}(x - c\lambda_t), \qquad t \to \infty, \tag{4.1}$$

holds uniformly for all $x \geq \gamma\lambda_t$ *for every fixed* $\gamma > c$ *under one of the following two conditions:*

(i) When $c \geq 0$*, relation (3.2) holds for some* $p > J_F^+$ *and all* $\delta > 0$*;*

(ii) When $c < 0$*, the relation*

$$\Pr(N_t \leq (1-\delta)\lambda_t) = o\left(\lambda_t \overline{F}(\lambda_t)\right), \qquad t \to \infty, \tag{4.2}$$

holds for all $0 < \delta < 1$*.*

Proof. Throughout this proof, every limiting relationship holds uniformly for all $x \geq \gamma\lambda_t$ as $t \to \infty$, as required by the theorem. Our starting point is the decomposition

$$\begin{aligned} \Pr(S_{N_t,c} > x) &= \left(\sum_{n<(1-\delta)\lambda_t} + \sum_{(1-\delta)\lambda_t \leq n \leq (1+\delta)\lambda_t} + \sum_{n>(1+\delta)\lambda_t} \right) \Pr(S_n > x - cn) \Pr(N_t = n) \\ &\hat{=} I_1(x,t) + I_2(x,t) + I_3(x,t), \end{aligned} \tag{4.3}$$

where $0 < \delta < 1$ is an arbitrarily fixed number to be specified later. We formulate the remaining proof into two parts according to $c \geq 0$ and $c < 0$, respectively.

To prove the result under condition (i) with $c \geq 0$, we follow the approach of Klüppelberg and Mikosch (1997) but in a much simpler manner. Recall that condition (3.2) implies (3.3), the weak law of large numbers of $\{N_t, t \geq 0\}$. In (4.3), choose $0 < \delta < 1$ such that $c(1+\delta) < \gamma$.

Note that, in $I_1(x,t)$, $x - cn \geq x - c(1-\delta)\lambda_t \asymp x - c\lambda_t$. Thus, by (1.1) (or, more exactly, by Corollary 3.1 of Tang (2006)), one can easily check that

$$I_1(x,t) = O(1)\, \lambda_t \overline{F}(x - c\lambda_t) \sum_{n<(1-\delta)\lambda_t} \frac{n}{\lambda_t} \Pr(N_t = n) = o(1) \lambda_t \overline{F}(x - c\lambda_t), \tag{4.4}$$

where the last step is obtained by (3.3) and the dominated convergence theorem.

Again, applying (1.1) yields

$$
\begin{aligned}
I_2(x,t) &\sim \sum_{(1-\delta)\lambda_t\le n\le(1+\delta)\lambda_t} n\overline{F}(x-cn)\Pr(N_t=n)\\
&\le \lambda_t\overline{F}(x-c(1+\delta)\lambda_t)\sum_{(1-\delta)\lambda_t\le n\le(1+\delta)\lambda_t}\frac{n}{\lambda_t}\Pr(N_t=n)\\
&\sim \lambda_t\overline{F}(x-c(1+\delta)\lambda_t)\\
&\le \lambda_t\overline{F}\left(\left(1-\frac{c\delta}{\gamma-c}\right)(x-c\lambda_t)\right), \qquad (4.5)
\end{aligned}
$$

where the last but one step is obtained by (3.3) and the dominated convergence theorem, and the last step is due to $x\ge\gamma\lambda_t$ and $\gamma>c$. Symmetrically,

$$
I_2(x,t)\gtrsim\lambda_t\overline{F}\left(\left(1+\frac{c\delta}{\gamma-c}\right)(x-c\lambda_t)\right). \qquad (4.6)
$$

Finally, to deal with $I_3(x,t)$, we set $v=1/p$ in (2.4) for some $p>J_F^+\ge1$. Arbitrarily choose some $\varepsilon\in(0,1)$, and split $I_3(x,t)$ into two parts as

$$
\begin{aligned}
I_3(x,t) &= \left(\sum_{(1-\varepsilon)x/c\ge n>(1+\delta)\lambda_t}+\sum_{n>((1+\delta)\lambda_t)\vee((1-\varepsilon)x/c)}\right)\Pr(S_n>x-cn)\Pr(N_t=n)\\
&= I_{31}(x,t)+I_{32}(x,t),
\end{aligned}
$$

where $I_{31}(x,t)$ is understood as 0 in case $(1+\delta)\lambda_t>(1-\varepsilon)x/c$. Since $x-cn$ is always positive in $I_{31}(x,t)$, it follows from Lemma 2.3, relations (2.1) and (3.2) that

$$
\begin{aligned}
I_{31}(x,t) &\le \sum_{(1-\varepsilon)x/c\ge n>(1+\delta)\lambda_t}\left(n\overline{F}(x/p)+C(n/x)^p\right)\Pr(N_t=n)\\
&= O(1)\overline{F}(x)\mathrm{E}N_t1_{(N_t>(1+\delta)\lambda_t)}+O(1)x^{-p}\mathrm{E}N_t^p1_{(N_t>(1+\delta)\lambda_t)}\\
&= o(1)\overline{F}(x-c\lambda_t)\mathrm{E}N_t^p1_{(N_t>(1+\delta)\lambda_t)}\\
&= o(1)\lambda_t\overline{F}(x-c\lambda_t).
\end{aligned}
$$

For $I_{32}(x,t)$, similarly as above, we have

$$
\begin{aligned}
I_{32}(x,t) &\le \sum_{n>((1+\delta)\lambda_t)\vee((1-\varepsilon)x/c)}\Pr(N_t=n)\\
&\le \sum_{n>((1+\delta)\lambda_t)\vee((1-\varepsilon)x/c)}\frac{n^p}{((1-\varepsilon)x/c)^p}\Pr(N_t=n)\\
&= O(1)x^{-p}\mathrm{E}N_t^p1_{(N_t>(1+\delta)\lambda_t)}\\
&= o(1)\lambda_t\overline{F}(x-c\lambda_t).
\end{aligned}
$$

Hence, it follows that

$$
I_3(x,t)=o(1)\lambda_t\overline{F}(x-c\lambda_t). \qquad (4.7)
$$

Substituting (4.4), (4.5), (4.6), and (4.7) into (4.3) yields

$$\lambda_t \overline{F}\left(\left(1+\frac{c\delta}{\gamma-c}\right)(x-c\lambda_t)\right) \lesssim \Pr\left(S_{N_t,c}>x\right) \lesssim \lambda_t \overline{F}\left(\left(1-\frac{c\delta}{\gamma-c}\right)(x-c\lambda_t)\right).$$

Therefore, relation (4.1) holds by the condition $F \in \mathcal{C}$ and the arbitrariness of δ.

We now switch our attention to the proof under condition (ii) with $c<0$. Without loss of generality, we assume $\gamma \in (c,0)$. Since $\lambda_t \overline{F}(\lambda_t) \to 0$, by Lemma 2.1 and condition (4.2), $\{N_t, t \geq 0\}$ fulfills (3.3), the weak law of large numbers. Again, in (4.3), we choose $0<\delta<1$ such that $c<\gamma(1+\delta)^{-1}<0$.

To deal with $I_1(x,t)$, with some $\widetilde{\gamma}>0$ we split the x-region into two disjoint regions as

$$[\gamma\lambda_t, \infty) = [\widetilde{\gamma}\lambda_t, \infty) \cup [\gamma\lambda_t, \widetilde{\gamma}\lambda_t).$$

For the first x-region $x \geq \widetilde{\gamma}\lambda_t$, by (1.1), it holds uniformly for all $x \geq \widetilde{\gamma}\lambda_t$ that

$$\begin{aligned}
I_1(x,t) &= O(1) \sum_{n<(1-\delta)\lambda_t} n\overline{F}(x-cn)\Pr\left(N_t=n\right) \\
&= O(1)\lambda_t \overline{F}(x)\Pr\left(N_t \leq (1-\delta)\lambda_t\right) \\
&= O(1)\lambda_t \overline{F}(x-c\lambda_t)\Pr\left(N_t \leq (1-\delta)\lambda_t\right) \\
&= o(1)\lambda_t \overline{F}(x-c\lambda_t),
\end{aligned}$$

where the last but one step can be verified as

$$\overline{F}(x-c\lambda_t) \geq \overline{F}\left((1-c/\widetilde{\gamma})x\right) \asymp \overline{F}(x).$$

For the second x-region $\gamma\lambda_t \leq x < \widetilde{\gamma}\lambda_t$, note that

$$\overline{F}(x-c\lambda_t) \geq \overline{F}((\tilde{\gamma}-c)\lambda_t) \asymp \overline{F}(\lambda_t).$$

Hence, by (4.2), we still have

$$I_1(x,t) \leq \Pr\left(N_t \leq (1-\delta)\lambda_t\right) = o(1)\lambda_t \overline{F}(x-c\lambda_t)$$

uniformly for all $\gamma\lambda_t \leq x < \tilde{\gamma}\lambda_t$. As a result, the relation

$$I_1(x,t) = o(1)\lambda_t \overline{F}(x-c\lambda_t) \tag{4.8}$$

holds uniformly for all $x \geq \gamma\lambda_t$.

For $I_2(x,t)$, note that, since $\gamma<0$, the variables x and n satisfy $x-cn \geq \gamma\lambda_t - cn \geq \left(\gamma(1-\delta)^{-1}-c\right)n$. It follows from (1.1) that

$$I_2(x,t) \sim \sum_{(1-\delta)\lambda_t \leq n \leq (1+\delta)\lambda_t} n\overline{F}(x-cn)\Pr\left(N_t=n\right).$$

Consequently, by (3.3), we obtain

$$\begin{aligned} I_2(x,t) &\lesssim (1+\delta)\lambda_t\overline{F}\left(x-(1-\delta)c\lambda_t\right)\Pr\left(1-\delta\le\frac{N_t}{\lambda_t}\le 1+\delta\right) \\ &\sim (1+\delta)\lambda_t\overline{F}\left(x-(1-\delta)c\lambda_t\right) \\ &\le (1+\delta)\lambda_t\overline{F}\left(\left(1+\frac{\delta c}{\gamma-c}\right)(x-c\lambda_t)\right). \end{aligned} \tag{4.9}$$

Symmetrically,

$$I_2(x,t)\gtrsim(1-\delta)\lambda_t\overline{F}\left(\left(1-\frac{\delta c}{\gamma-c}\right)(x-c\lambda_t)\right). \tag{4.10}$$

Finally, in $I_3(x,t)$, the variables x and n satisfy $x-cn\ge\gamma\lambda_t-cn\ge\left(\gamma(1+\delta)^{-1}-c\right)n$. Therefore, by (1.1),

$$I_3(x,t)\sim\sum_{n>(1+\delta)\lambda_t}n\overline{F}(x-cn)\Pr\left(N_t=n\right)\le\overline{F}\left(x-c\lambda_t\right)\mathrm{E}N_t 1_{(N_t>(1+\delta)\lambda_t)}.$$

It follows from (3.3) and Lemma 2.1 that

$$I_3(x,t)=o(1)\lambda_t\overline{F}(x-c\lambda_t). \tag{4.11}$$

Directly substituting (4.8), (4.9), (4.10), and (4.11) into (4.3) yields

$$(1-\delta)\lambda_t\overline{F}\left(\left(1-\frac{\delta c}{\gamma-c}\right)(x-c\lambda_t)\right)\lesssim\Pr\left(S_{N_t,c}>x\right)\lesssim(1+\delta)\lambda_t\overline{F}\left(\left(1+\frac{\delta c}{\gamma-c}\right)(x-c\lambda_t)\right).$$

Therefore, relation (4.1) still holds by the condition $F\in\mathcal{C}$ and the arbitrariness of δ. □

5 On a quasi-renewal model

As an important special case of Theorem 4.1, we establish a result of precise large deviations for a nonstandard renewal model in which innovations, modelled as real-valued random variables, are END and identically distributed, while their inter-arrival times are ND, identically distributed, and independent of the innovations.

Theorem 5.1. *Let $\{X_k, k=1,2,\ldots\}$ be a sequence of END random variables with common distribution $F\in\mathcal{C}$, zero mean, and satisfying (1.2). Let $\{N_t, t\ge 0\}$ be a quasi-renewal counting process defined by*

$$N_t=\#\left\{n=1,2,\ldots:\sum_{k=1}^{n}Y_k\le t\right\},\qquad t\ge 0, \tag{5.1}$$

where $\{Y_k, k=1,2,\ldots\}$, independent of $\{X_k, k=1,2,\ldots\}$, form a sequence of nonnegative random variables with common distribution G non-degenerate at zero. Then, the precise large deviation result (4.1) holds under one of the following two conditions:

(i) When $c \geq 0$, $\{Y_k, k = 1, 2, \ldots\}$ are LND with common finite mean;

(ii) When $c < 0$, $\{Y_k, k = 1, 2, \ldots\}$ are ND and $\overline{G}(x) = o\left(\overline{F}(x)\right)$ as $x \to \infty$.

According to Theorem 4.1, it suffices to verify conditions (3.2) and (4.2) for cases (i) and (ii), respectively. We formulate this verification into a proposition below.

Proposition 5.1. *Let $\{N_t, t \geq 0\}$ be a quasi-renewal counting process defined by (5.1), in which $\{Y_k, k = 1, 2, \ldots\}$ form a sequence of nonnegative random variables with common distribution G and finite mean $1/\lambda > 0$. We have the following two results:*

(i) If $\{Y_k, k = 1, 2, \ldots\}$ are LND, then relation (3.2) holds for every $p > 0$ and $\delta > 0$;

(ii) If $\{Y_k, k = 1, 2, \ldots\}$ are ND and $\mathrm{E}Y_1^r < \infty$ for some $r > 1$, then, for every fixed $0 < \delta < 1$ and $p > 0$, there exist positive numbers v and C such that, for all $n = 1, 2, \ldots$ and $x \geq \gamma n$,

$$\Pr\left(N_t \leq (1-\delta)\lambda_t\right) \leq (1-\delta)\lambda_t \overline{G}\left(v\lambda_t\right) + C\lambda_t^{-p}. \tag{5.2}$$

Proof. Note that, if the random variables $\{Y_k, k = 1, 2, \ldots\}$ are either LND or UND, they are pairwise ND. Under the pairwise ND structure, Matuła (1992) established the strong law of large numbers,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} Y_k = \frac{1}{\lambda}, \qquad \text{almost surely.}$$

It follows that, for every $0 < \varepsilon < 1$,

$$\begin{aligned}
\Pr\left(\left|\frac{N_t}{\lambda t} - 1\right| \geq \varepsilon\right) &= \Pr\left(N_t \geq (1+\varepsilon)\lambda t\right) + \Pr\left(N_t \leq (1-\varepsilon)\lambda t\right) \\
&= \Pr\left(\sum_{1\leq k\leq (1+\varepsilon)\lambda t} Y_k \leq t\right) + \Pr\left(\sum_{1\leq k\leq (1-\varepsilon)\lambda t} Y_k \geq t\right) \\
&\to 0, \qquad t \to \infty.
\end{aligned}$$

Hence, $N_t/(\lambda t) \xrightarrow{P} 1$ as $t \to \infty$.

(i) By this last limiting result and the LND structure of $\{Y_k, k = 1, 2, \ldots\}$, one can follow the proof of Lemma 3.5 of Tang et al. (2001) and use Lemma 2.2 of Tang (2006) to show that, for every fixed $p > 0$ and $\delta > 0$,

$$\sum_{n > (1+\delta)\lambda t} n^p \Pr\left(N_t \geq n\right) = o(1), \qquad t \to \infty.$$

Using the above relation with $p = 1$ and the dominated convergence theorem, we obtain

$$\lim_{t\to\infty} \frac{\mathrm{E}N_t}{\lambda t} = \lim_{t\to\infty} \mathrm{E}\frac{N_t}{\lambda t} 1_{(N_t \leq (1+\delta)\lambda t)} = 1;$$

that is, $\lambda_t \sim \lambda t$ as $t \to \infty$.

(ii) With the ND structure of $\{Y_k, k = 1, 2, \ldots\}$, the relation $\lambda_t \sim \lambda t$ as $t \to \infty$ still holds. For arbitrarily fixed $p > 0$ and $0 < \delta < 1$, Lemma 2.2 implies that, for some $\widetilde{v} > 0$ and $\widetilde{C} > 0$,

$$\Pr\left(N_t \leq (1-\delta)\lambda_t\right) = \Pr\left(\sum_{1\leq k\leq (1-\delta)\lambda_t} Y_k \geq t\right) \leq (1-\delta)\lambda_t \overline{G}\left(\widetilde{v}t\right) + \widetilde{C}t^{-p}.$$

Hence, relation (5.2) holds with suitably chosen $v > 0$ and $C > 0$. □

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