Extinction Probability of Interacting Branching Collision Processes

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Abstract

We consider the uniqueness and extinction properties of the Interacting Branching Collision Process (IBCP), which consists of two strongly interacting components: an ordinary Markov branching process (MBP) and a collision branching process (CBP). We establish that there is a unique IBCP, and derive necessary and sufficient conditions for it to be non-explosive that are easily to be checked. Explicit expressions are obtained for the extinction probabilities for both regular and irregular cases. The associated expected hitting times are also considered. Examples are provided to illustrate our results.

Keywords: Markov branching processes; Collision branching processes; Interaction; Regularity; Uniqueness; Extinction Probability; Extinction time.

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1. Introduction

The primary aim of this paper is to tackle a very interesting as well as challenging open problem regarding the extinction probability of the interacting branching collision process (defined below) which is an important class of interacting branching systems. As is well-known, there has been an extensive interest in generalizing the ordinary Markov branching processes (MBPs) into more general interacting branching models. Such increasing interest is mainly due to the fact that the basic property which governs the evolution of an MBP (i.e. different particles act independently) is not appropriate in many realistic situations. In-

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deed, in realistic situations, particularly in biological science, individuals (particles) usually interact with each other.

The interacting branching collision process, as one of the most important sub-class of interacting branching systems, consists of two strongly interacting components. The first component is an ordinary Markov branching process (MBP) and the other is a collision branching process (CBP). For the former see the good references of Harris [8], Athreya and Ney [4], Asmussen and Hering [2], and Athreya and Jagers [3] whilst for the latter, see Kalinkin [9, 10], Chen et al [5, 6], Lange [11] and the references therein. Note that comparing with the huge publications for the former there exist much fewer papers in the literature to discuss the latter. This is because in the evolution of a collision branching process, different from an MBP, the branching events are effected by the interaction/collision of *pairs* of particles, rather than by the particles individually as in an MBP and thus the analysis becomes much more difficult. However, many challenging but important and interesting questions have arisen due to such interaction effect.

The interacting branching collision process is, however, even more challenging and interesting since in addition to the interaction within the CBP, the two components also strongly interact with each other. Hence to investigate the properties of this model is of great significance.

Although few progresses have been made even until now, this model has been attracted much attention. The interests toward this model can be traced back at least to 1982, see Sevastyanov and Kalinkin [13]. One of the important questions, the extinction probabilities have been addressed in Kalinkin [10] and an explicit expression for extinction probabilities has been obtained for some special examples. More importantly, some new methods and techniques in tackling these questions have been introduced and applied in Kalinkin [9, 10]. In particular, in addition to use the Kolmogorov forward equation, the Kolmogorov backward equation and the exponential generating function were used in Kalinkin [9, 10]. Some further discussions could be seen in Lange [11].

The main aim of this paper is further to consider more general models for such interacting branching collision process based on the previous research as mentioned above. We shall focus on discussing the two most basic questions, the uniqueness and extinction probabilities, for the general model and resolve some related interesting and important open questions. Our main method will be using the Kolmogorov forward equation.

The general model we shall address in this paper is a continuous time Markov chain defined on the state space of nonnegative integers $\mathbf{Z}_+ = \{0, 1, 2, \cdots\}$ that represents the evolution of some interacting particles. More formally we define the model by specifying its infinitesimal characteristic, i.e., the so-called q-matrix as follows.

Definition 1.1. A q-matrix $Q = (q_{ij};\ i, j \in \mathbf{Z}_+)$ is called an interacting branching-collision q-matrix (henceforth referred to as an IBC q-matrix), if

$$q_{ij} = \begin{cases} \binom{i}{2} c_{j-i+2} + i b_{j-i+1}, & \text{if } i \geq 1, j \geq i-2 \\ 0, & \text{otherwise} \end{cases} \tag{1.1}$$

where

$$\begin{cases} c_0 > 0, c_j \geq 0 \ (j \neq 2), \sum_{k=3}^{\infty} c_k > 0, \ 0 < \sum_{j\neq 2} c_j = -c_2 < \infty \\ b_0 > 0, b_j \geq 0 \ (j \neq 1), \sum_{k=2}^{\infty} b_k > 0, \ 0 < \sum_{j\neq 1} b_j = -b_1 < \infty \end{cases} \tag{1.2}$$

together with the conventions $b_{-1} = 0$ and $\binom{1}{2} = 0$.

Definition 1.2. A Markov interacting branching-collision process (henceforth referred to as an IBCP) is a continuous-time Markov chain on the state space $\mathbf{Z}_+$ whose transition function $P(t) = (p_{ij}(t); i, j \in \mathbf{Z}_+)$ satisfies

$$P'(t) = P(t)Q \tag{1.3}$$

where Q is given in (1.1) − (1.2).

The structure of this paper is as follows. Some preliminary results are firstly obtained in Section 2. Uniqueness and regularity criteria are then obtained in Section 3. We show that the IBCP is honest i.e. the infinitesimal q-matrix Q is regular, if and only if the mean birth rate is less than or equal to the mean death rate for the CBP component only. We also show that there always exists only one IBCP for a given q-matrix Q. The important question of extinction probability together with the mean extinction time is extensively discussed in Sections 4 and 5. The regular case, which is relatively easy, is fully discussed in Section 4 while the very subtle irregular, i.e. the explosive, case is deeply analysed in Section 5. For both cases, the explicit expressions for extinction probabilities are presented. We show that the extinction probabilities will be mainly dominated by the CBP component rather than the MBP component. In the final Section 6, a couple of examples are provided to illustrate the results obtained in the previous sections.

2. Preliminary

In order to investigate properties of IBCPs, it is necessary to define the generating functions of the two known sequences $\{c_k; k \geq 0\}$ and $\{b_k; \geq 0\}$ as

$$C(s) = \sum_{k=0}^{\infty} c_k s^k \quad \text{and} \quad B(s) = \sum_{k=0}^{\infty} b_k s^k. \tag{2.1}$$

These two functions play extremely important role in our later analysis. It is clear that $C(s)$ and $B(s)$ are well defined at least on $[-1, 1]$. The following simple yet important properties of these functions will be constantly used in this paper and we state them here for the sake of convenience. However, their proofs are easy and well-known and thus omitted.

Lemma 2.1. (i) *The equation $C(s) = 0$ has at most two roots in $[0, 1]$ and exactly one root in $[-1, 0)$. More specifically, if $C'(1) \leq 0$ then $C(s) > 0$ for all $s \in [0, 1)$ and 1 is the only root of the equation $C(s) = 0$ in $[0, 1]$, which is simple or with multiplicity 2 according to $C'(1) < 0$ or $C'(1) = 0$, while if $0 < C'(1) \leq +\infty$ then $C(s) = 0$ has an additional simple root ρ_c satisfying $0 < \rho_c < 1$ such that $C(s) > 0$ for $s \in (0, \rho_c)$ and $C(s) < 0$ for $s \in (\rho_c, 1)$. Also $C(s) = 0$ has exactly one root, denoted by ζ_c, in $[-1, 0]$*

such that $C(s) > 0$ for all $s \in (\zeta_c, 0]$ and $|\zeta_c| \le \rho_c$. This root is simple unless $C'(1) = 0$ and $\sum_{k=0}^{\infty} c_{2k+1} = 0$. Also, $|\zeta_c| = \rho_c$ if and only if $\sum_{k=0}^{\infty} c_{2k+1} = 0$. Moreover, $C(z) = 0$ has no other root in the complex disk $\{z; |z| \le 1\}$.

(ii) *The equation $B(s) = 0$ has at most two roots in $[0, 1]$. More specifically, if $B'(1) \le 0$ then $B(s) > 0$ for all $s \in [-1, 1)$ and 1 is the only root of $B(s) = 0$ in $[0, 1)$. If $0 < B'(1) \le +\infty$ then $B(s) = 0$ has an additional root in $[0, 1)$, denoted by ρ_b, such that $B(s) > 0$ for all $s \in [-1, \rho_b)$ and $B(s) < 0$ for $s \in (\rho_b, 1)$. Moreover, $B(z) = 0$ has no other root in the complex disk $\{z; |z| \le 1\}$.*

Throughout this paper, we shall let ρ_c and ρ_b denote the smallest nonnegative root of $C(s) = 0$ and $B(s) = 0$ respectively.

Lemma 2.2. *Suppose that Q is an IBC q-matrix as defined in (1.1) − (1.2) and let $P(t) = (p_{ij}(t); i, j \ge 0)$ and $\Phi(\lambda) = (\phi_{ij}(\lambda); i, j \ge 0)$ be a Q-function and its Q-resolvent, respectively. Further assume that the Q-function $P(t)$ and Q-resolvent $\Phi(\lambda)$ satisfy the Kolmogorov forward equation (1.3). Then for any $i \ge 0, t \ge 0, \lambda > 0$ and $|s| < 1$, we have*

$$\frac{\partial F_i(t,s)}{\partial t} = \frac{C(s)}{2} \cdot \frac{\partial^2 F_i(t,s)}{\partial s^2} + B(s) \cdot \frac{\partial F_i(t,s)}{\partial s} \tag{2.2}$$

or equivalently,

$$\Phi_i(\lambda, s) - s^i = \frac{C(s)}{2} \cdot \frac{\partial^2 \Phi_i(\lambda,s)}{\partial s^2} + B(s) \cdot \frac{\partial \Phi_i(\lambda,s)}{\partial s} \tag{2.3}$$

where $F_i(t,s) = \sum_{j=0}^{\infty} p_{ij}(t)s^j$ and $\Phi_i(\lambda,s) = \sum_{j=0}^{\infty} \phi_{ij}(\lambda)s^j$.

Proof. It follows from the Kolmogorov forward equation (1.3) that for any $i, j \ge 0$,

$$p'_{ij}(t) = \sum_{k=2}^{j+2} p_{ik}(t)\binom{k}{2}c_{j-k+2} + \sum_{k=1}^{j+1} p_{ik}(t)kb_{j-k+1}.$$

Multiplying s^j on both sides of the above equality and summing over $\mathbf{Z}_+$ we immediately obtain (2.2). Taking Laplace transform on both sides of (2.2) then yields (2.3). □

Lemma 2.3. *Suppose that Q is an IBC q-matrix as defined in (1.1) − (1.2). Let $P(t) = (p_{ij}(t); i, j \ge 0)$ be a Q-function that satisfies the Kolmogorov forward equations. Then*

(i) *$\int_0^{\infty} p_{ij}(t)dt < +\infty$ $(i, j \ge 1)$ and thus $\lim_{t\to\infty} p_{ij}(t) = 0$ $(i, j \ge 1)$.*

(ii) *For any $i \ge 1$ and $s \in [0, 1)$,*

$$\sum_{j=1}^{\infty} (\int_0^{\infty} p_{ij}(t)dt) \cdot s^j < +\infty. \tag{2.4}$$

Proof. It is clear that every positive state is transient and thus (i) follows. This simple fact can also be easily obtained analytically. Indeed, by Kolmogorov forward equation we have

$$p'_{i0}(t) = p_{i2}(t)c_0 + p_{i1}(t)b_0, \quad i \ge 1$$

which implies that $\int_0^\infty p_{i2}(t)dt < +\infty$ since $c_0 > 0$. Hence, by the irreducibility of positive states we know that $\int_0^\infty p_{ij}(t)dt < +\infty$ for all $i, j \geq 1$.

We now prove (2.4). First note that the partial differential equation (2.2) is just saying that for any $i \geq 1$, $|s| < 1$, we have

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j = \frac{C(s)}{2} \cdot \sum_{k=2}^{\infty} p_{ik}(t)k(k-1)s^{k-2} + B(s)\sum_{k=1}^{\infty} p_{ik}(t)ks^{k-1} \tag{2.5}$$

which can be rewritten as

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j = \sum_{k=1}^{\infty}[\frac{(k-1)C(s)}{2} + sB(s)]p_{ik}(t)ks^{k-2}. \tag{2.6}$$

Now, if $C'(1) \leq 0$, then by Lemma 2.1 we have that $C(\hat{s}) > 0$ for any $\hat{s} \in [0, 1)$. It follows that there exists a $\hat{k} \geq 2$ such that $\frac{(k-1)C(\hat{s})}{2} + \hat{s}B(\hat{s}) > 0$ for any $k \geq \hat{k}$. Then by (2.6) we obtain

$$[\frac{(\hat{k}-1)C(\hat{s})}{2} + \hat{s}B(\hat{s})] \cdot \sum_{k=\hat{k}}^{\infty} p_{ik}(t)k\hat{s}^{k-2} \leq \sum_{j=0}^{\infty} p'_{ij}(t)\hat{s}^j - \sum_{k=1}^{\hat{k}-1}[\frac{(k-1)C(\hat{s})}{2} + \hat{s}B(\hat{s})]p_{ik}(t)k\hat{s}^{k-2}.$$

Integrating the above inequality yields that

$$\begin{aligned} & [\frac{(\hat{k}-1)C(\hat{s})}{2} + \hat{s}B(\hat{s})] \cdot \sum_{k=\hat{k}}^{\infty}(\int_0^\infty p_{ik}(t)dt)k\hat{s}^{k-2} \\ \leq & \lim_{t\to\infty} p_{i0}(t) - \hat{s}^i - \sum_{k=1}^{\hat{k}-1}[\frac{(k-1)C(\hat{s})}{2} + \hat{s}B(\hat{s})](\int_0^\infty p_{ik}(t)dt)k\hat{s}^{k-2} \\ < & +\infty \end{aligned}$$

which implies (2.4).

On the other hand, if $0 < C'(1) \leq +\infty$, then again by Lemma 2.1 we know that $C(s) = 0$ has a smallest nonnegative root $\rho_c \in [0, 1)$ such that $C(s) < 0$ for any $s \in (\rho_c, 1)$. Now, for any $\hat{s} \in (\rho_c, 1)$, there exists a $\hat{k} \geq 2$ such that $\frac{(k-1)C(\hat{s})}{2} + \hat{s}B(\hat{s}) < 0$ for any $k \geq \hat{k}$. Then again by (2.6), this time we get that

$$[\frac{(\hat{k}-1)C(\hat{s})}{2} + \hat{s}B(\hat{s})] \cdot \sum_{k=\hat{k}}^{\infty} p_{ik}(t)k\hat{s}^{k-2} \geq \sum_{j=0}^{\infty} p'_{ij}(t)\hat{s}^j - \sum_{k=1}^{\hat{k}-1}[\frac{(k-1)C(\hat{s})}{2} + \hat{s}B(\hat{s})]p_{ik}(t)k\hat{s}^{k-2}.$$

Integrating the above inequality yields that

$$\begin{aligned} & [\frac{(\hat{k}-1)C(\hat{s})}{2} + \hat{s}B(\hat{s})] \cdot \sum_{k=\hat{k}}^{\infty}(\int_0^\infty p_{ik}(t)dt)k\hat{s}^{k-2} \\ \geq & \lim_{t\to\infty} p_{i0}(t) - \hat{s}^i - \sum_{k=1}^{\hat{k}-1}[\frac{(k-1)C(\hat{s})}{2} + \hat{s}B(\hat{s})](\int_0^\infty p_{ik}(t)dt)k\hat{s}^{k-2} \\ > & -\infty \end{aligned}$$

which implies (2.4) since $\frac{(\hat{k}-1)C(\hat{s})}{2}+\hat{s}B(\hat{s})<0$. The proof is complete. □

3. Uniqueness

In order to discuss the regularity and uniqueness, we also need the following result.

Lemma 3.1. *Suppose that $Q=(q_{ij};i,j\ge 0)$ is a conservative q-matrix and $\bar{k}\ge 1$ is an integer. Define a new matrix $Q^*=(q^*_{ij};i,j\ge 0)$ as*

$$q^*_{ij}=\begin{cases} q_{ij}, & \text{if } i>\bar{k} \\ 0, & \text{otherwise.}\end{cases}$$

Then Q^* is also a a conservative q-matrix. Moreover, if Q is regular then so is Q^*.

Proof. We only need to prove the last conclusion. Suppose that Q^* is not regular. Then the equation

$$Q^*Y\ge\lambda Y$$

has a nontrivial nonnegative and bounded solution for some $\lambda>0$, denoted by $Y=(y_i;i\ge 0)$. It is easily seen that $y_i=0$ for $i\le\bar{k}$. We claim that $Y=(y_i;i\ge 0)$ is also a solution of

$$QY\ge\lambda Y.$$

Indeed, for $i\le\bar{k}$,

$$(QY)_i=\sum_{j=0}^{\infty}q_{ij}y_j=\sum_{j=\bar{k}+1}^{\infty}q_{ij}y_j\ge 0=\lambda y_i$$

since $y_i=0$ for all $i\le\bar{k}$. For $i>\bar{k}$,

$$(QY)_i=(Q^*Y)_i\ge\lambda y_i.$$

Therefore, Q is not regular. The proof is complete. □

In this section, we consider regularity and uniqueness for IBCPs. For convenience, from now on, we shall always assume that $C'(1)+B'(1)<+\infty$ for any given IBC q-matrix Q.

Theorem 3.1. *Let Q be an IBC q-matrix as defined in (1.1) − (1.2). Then Q is regular if and only if $C'(1)\le 0$.*

Proof. First assume that $C'(1)\le 0$. If we further assume that $B'(1)\le 0$, then both $C(s)$ and $B(s)$ are positive for all $s\in[0,1)$. It follows from (2.4) that

$$\lambda\sum_{j=0}^{\infty}\phi_{ij}(\lambda)s^j\ge s^i,\quad s\in[0,1). \tag{3.1}$$

Letting $s\uparrow 1$ in (3.1) yields that $\lambda\sum_{j=0}^{\infty}\phi_{ij}(\lambda)=1$, i.e., Q is regular.

Now suppose $C'(1) \le 0$ and $B'(1) > 0$. Since $B'(1) < +\infty$ and $C'(1) \le 0$, we obtain from (2.3) that

$$\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j - s^i \ge B(s) \sum_{k=1}^{\infty} \phi_{ik}(\lambda) k s^{k-1}, \quad s \in [0, 1). \tag{3.2}$$

If Q is not regular, then there exists an $i \ge 0$ and a $\lambda > 0$ such that $\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) < 1$. Hence, there exist a $\delta > 0$ and an $\tilde{s} \in (\rho_b, 1)$ such that for all $s \in [\tilde{s}, 1]$ we have

$$s^i - \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j > \delta. \tag{3.3}$$

Noting the fact that $B(s) < 0$ for all $\tilde{s} \in (\rho_b, 1)$ and thus by using (3.2) and (3.3) we obtain

$$\sum_{k=1}^{\infty} \phi_{ik}(\lambda) k s^{k-1} \ge \frac{s^i - \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j}{-B(s)} \ge \frac{\delta}{-B(s)}, \quad s \in [\tilde{s}, 1). \tag{3.4}$$

Therefore,

$$\sum_{k=1}^{\infty} \phi_{ik}(\lambda)(1 - \tilde{s}^k) \ge \int_{\tilde{s}}^{1} \frac{\delta}{-B(s)} ds = +\infty$$

which is a contradiction and hence Q is regular.

We now prove the converse. Suppose that $C'(1) > 0$. By a similar argument as in Chen et.al [5], we could find two constants a^* and b^* such that

$$2c_0 + c_1 < a^* < b^* < \sum_{j=1}^{\infty} j c_{j+2} \tag{3.5}$$

and

$$\sum_{j=1}^{\infty} c_{j+2} \sum_{k=1}^{j} \left(\frac{a^*}{b^*} \right)^{k-1} > b^*. \tag{3.6}$$

Now, we choose an $\varepsilon \in (0, b^* - a^*)$ and let $i_0 = [\frac{2b_0}{\varepsilon}] + 1$ and then define a q-matrix $\tilde{Q} = (\tilde{q}_{ij}; i, j \ge 0)$ as

$$\tilde{q}_{ij} = \begin{cases} \binom{i}{2} c_{j-i+2} + i b_{j-i+1}, & \text{if } i > i_0, j \ge i - 2 \\ 0, & \text{otherwise.} \end{cases}$$

By Lemma 3.1, we only need to prove that $\tilde{Q}$ is not regular. For this purpose, define a (conservative) birth-death q-matrix $Q^* = (q^*_{ij}; i, j \in Z_+)$ by

$$q^*_{ij} = \begin{cases} \binom{i}{2} b^* & \text{if } \ i > i_0, j = i + 1, \ i \ge 2 \\ \binom{i}{2} (a^* + \varepsilon) & \text{if } \ i > i_0, j = i - 1, \ i \ge 2 \\ -\binom{i}{2} (b^* + a^* + \varepsilon) & \text{if } \ j = i > i_0 \\ 0 & \text{otherwise.} \end{cases}$$

Since $b^* > a^* + \varepsilon > 0$ and $\sum_{i=2}^{\infty} \binom{i}{2}^{-1} < +\infty$, it is easy to see that Q^* is not regular. Hence, the equation

$$(\lambda I - Q^*)u = 0 \qquad (\lambda > 0) \tag{3.7}$$

has a non-trivial (non-negative) bounded solution, denoted by $u^* = (u_i,\ i \ge 0)$. Clearly $u_i > 0$ for all $i > i_0$. It is also easy to see that $u_0 = \cdots = u_{i_0} = 0$ and

$$b^*(u_{i+1} - u_i) = (a^* + \varepsilon)(u_i - u_{i-1}) + \lambda u_i \binom{i}{2}^{-1}, \qquad i > i_0. \tag{3.8}$$

In particular, for $i = i_0 + 1$ we have $b^*(u_{i_0+2} - u_{i_0+1}) = (a^* + \varepsilon + \lambda)u_{i_0+1}$ (> 0)which implies that $(u_i; i > i_0)$ is strictly increasing in i. From (3.8) it is easily seen that, for all $k \ge 1$ and $i > i_0$,

$$u_{i+k} - u_{i+k-1} \ge \left(\frac{a^* + \varepsilon}{b^*}\right)^{k-1} (u_{i+1} - u_i) > \left(\frac{a^*}{b^*}\right)^{k-1} (u_{i+1} - u_i) \tag{3.9}$$

and

$$u_{i-1} - u_{i-2} \le \left(\frac{b^*}{a^* + \varepsilon}\right) (u_i - u_{i-1}) < \left(\frac{b^*}{a^*}\right) (u_i - u_{i-1}). \tag{3.10}$$

Now, for $i > i_0$, we have

$$\begin{aligned}
(\tilde{Q}u)_i &= \binom{i}{2}\left(c_0(u_{i-2} - u_i) + c_1(u_{i-1} - u_i) + \sum_{j=i+1}^{\infty} c_{j-i+2}(u_j - u_i)\right) \\
&\quad + i[b_0(u_{i-1} - u_i) + \sum_{j=i+1}^{\infty} b_{j-i+1}(u_j - u_i)] \\
&= \binom{i}{2}(-I_d + I_b) + i(-J_d + J_b)
\end{aligned}$$

where I_d, I_b, J_d and J_b should be self-explained by the above.

Now by (3.8) and (3.9), we get that

$$I_b \ge \sum_{j=1}^{\infty} c_{j+2} \sum_{k=1}^{j} \left(\frac{a^*}{b^*}\right)^{k-1} (u_{i+1} - u_i) > b^*(u_{i+1} - u_i).$$

and that

$$J_b \ge \sum_{j=1}^{\infty} b_{j+1} \sum_{k=1}^{j} \left(\frac{a^*}{b^*}\right)^{k-1} (u_{i+1} - u_i) = \tilde{b}(u_{i+1} - u_i)$$

where $\tilde{b} = \sum_{j=1}^{\infty} b_{j+1} \sum_{k=1}^{j} \left(\frac{a^*}{b^*}\right)^{k-1}$.

Similarly, by (3.10) we have

$$I_d \leq \left(c_0\left(\frac{b^*}{a^*}\right)+(c_0+c_1)\right)(u_i-u_{i-1}) < a^*(u_i-u_{i-1}). \tag{3.11}$$

and $J_d = b_0(u_i - u_{i-1})$. Therefore,

$$\binom{i}{2}I_b + iJ_b \geq \binom{i}{2}b^*(u_{i+1}-u_i) + i\tilde{b}(u_{i+1}-u_i)$$

and

$$\begin{aligned}\binom{i}{2}I_d + iJ_d &\leq \binom{i}{2}a^*(u_{i+1}-u_i) + ib_0(u_{i+1}-u_i)\\ &= \binom{i}{2}(a^*+\varepsilon)(u_i-u_{i-1}) - [\binom{i}{2}\varepsilon - ib_0](u_i-u_{i-1})\end{aligned}$$

Therefore, $u^* = (u_i; i \geq 0)$ satisfies

$$\tilde{Q}u^* \geq \lambda u^*. \tag{3.12}$$

Indeed, (3.12) is obvious true for $i \leq i_0$. As to $i > i_0$, by using (3.10)−(3.11) and (3.7), we could easily obtain that

$$(\tilde{Q}u)_i \geq \lambda u_i + [\binom{i}{2}\varepsilon - ib_0](u_i - u_{i-1}) \geq \lambda u_i.$$

Thus $\tilde{Q}$ is not regular and hence by Lemma 3.1, Q is not regular. The proof is complete. □

Remark 3.1. Theorem 3.1 provides a regularity criterion under the assumption (see the beginning of this section) that $C'(1)+B'(1) < +\infty$. However, by checking the proof carefully, one will find that the conclusion still holds if $C'(1) = +\infty$ provided that $B'(1) < +\infty$. Even if this latter condition is removed, Q may still be regular. For example, we can prove that if $C'(1) \leq 0$, $B'(1) > 0$ and

$$\int_\gamma^1 \frac{1}{C(s)} e^{\int_0^s \frac{2B(x)}{C(x)}dx} ds = +\infty$$

for some (or equivalently, for all) $\gamma \in (\rho_b, 1)$, then Q is regular. Indeed, suppose that Q is not regular, then there exist ε and $\gamma \in (\rho_b, 1)$ such that $s^i - \lambda\Phi(s) > \varepsilon > 0$ for all $s \in (\gamma, 1)$. Therefore, by (2.5) we have

$$\Phi'(s)e^{\int_\gamma^s \frac{2B(x)}{C(x)}dx} - \Phi'(\gamma) \leq -2\varepsilon \int_\gamma^s \frac{1}{C(y)} e^{\int_\gamma^y \frac{2B(x)}{C(x)}dx} dy.$$

Letting $s \to 1$ in the above inequality yields that $\lim_{s\to 1} \Phi'(s)e^{\int_\gamma^s \frac{2B(x)}{C(x)}dx} = -\infty$ which is a contradiction. Hence, Q is regular.

We now turn to consider the uniqueness problem of Q-functions which satisfy the Kolmogorov forward equation.

Theorem 3.2. *There always exists exactly one Q-function that satisfies the Kolmogorov forward equation. That is that there always exists only one IBCP which is the Feller minimal process for any given Q.*

Proof. By Theorem 3.1, we only need to consider the case that $C'(1) > 0$. By Theorem 2.2.8 of Anderson [1], we only need to show that the equation

$$\lambda Y = YQ, Y \geq 0, Y \cdot \mathbf{1} < +\infty \tag{3.13}$$

has no nontrivial solution for some (and then for all) $\lambda > 0$, where $Y \cdot \mathbf{1}$ denotes the inner product of Y and the vector $\mathbf{1}$ whose components are all 1.

Suppose that $Y = (y_k; k \geq 0)$ is a nontrivial solution of (3.13) with $\lambda = 1$. Then $y_0 > 0$ and (3.13) can be written as

$$Y(s) = \frac{C(s)}{2} Y''(s) + B(s)Y'(s), \qquad |s| < 1 \tag{3.14}$$

where $Y(s) = \sum_{k=0}^{\infty} y_k s^k$.

First consider the case that $C'(1) > 0$. If $B'(1) > 0$ we have that both $C(s) < 0$ and $B(s) < 0$ for all $s \in (\rho_c \vee \rho_b, 1)$ and hence the right hand side of (3.14) is negative. However, the left hand side is positive which is a contradiction. If $B'(1) \leq 0$, then

$$Y(s) \leq B(s)Y'(s), \qquad s \in (\rho_c, 1).$$

Hence

$$\ln Y(1) - \ln Y(\rho_c) \geq \int_{\rho_c}^{1} \frac{ds}{B(s)} = +\infty$$

which contradicts with $0 < Y(1) < \infty$.

4. Extinction Probability: regular case

Let $\{X(t); t \geq 0\}$ be the unique interacting branching-collision process with a given IBC q-matrix Q as defined in (1.1) – (1.2) and let $P(t) = (p_{ij}(t); i, j \in \mathbf{Z}_+)$ be the unique Q-function. Let

$$\tau_0 = \inf\{t > 0; X(t) = 0\}$$

and

$$a_i = P(\tau_0 < \infty | X(0) = i), \quad i \geq 1,$$

be the extinction time and extinction probability, respectively.

We shall consider the absorbing behavior of IBCP in two different cases, regular and irregular, in this and the next section, respectively. As a preparation, we first provide two lemmas which hold true for both cases.

Denote

$$G_i(s) = \sum_{k=1}^{\infty}\left(\int_0^{\infty} p_{ik}(t)dt\right)s^k, \quad i \geq 1. \tag{4.1}$$

Then by (2.4) we know that $G_i(s)$ is well-defined for all $|s| < 1$.

In addition, define

$$H(y) = \int_0^y \frac{B(x)}{C(x)}dx, \quad y \in (\zeta_c, \rho_c) \tag{4.2}$$

where the integral should be taken along the inverse direction if $y < 0$. By Lemma 2.1, we know that $H(y)$ is finite for all $y \in (\zeta_c, \rho_c)$. Also let

$$A(y) = \exp\{2H(y)\}, \quad y \in (\zeta_c, \rho_c). \tag{4.3}$$

It is obvious that $H(0) = 0$ and $H(y) < 0$ if $y \in (\zeta_c, 0)$ and hence $A(0) = 1$ and $A(y) < 1$ if $y \in (\zeta_c, 0)$. It is also clear that $A(y) \to 0$ if and only if $H(y) \to -\infty$. The following simple lemma provides further information about the two functions $H(y)$ and $A(y)$ which will be useful in our later analysis.

Lemma 4.1. (i) $\lim_{y\to\zeta_c^+} H(y) = -\infty$ *and* $\lim_{y\to\zeta_c^+} A(y) = 0$. *Moreover, we have* $A(y) \sim K(y - \zeta_c)$ *when* $y \to \zeta_c^+$ *where* $0 < K < +\infty$ *and in fact,* $K = \frac{2B(\zeta_c)}{C'(\zeta_c)}$.

(ii) *Suppose that* $0 < C'(1) \leq +\infty$ *and thus* $\rho_c < 1$. *If* $\rho_b = \rho_c < 1$ *then* $0 \leq H(\rho_c) < +\infty$.
If $\rho_b < \rho_c < 1$ *then* $\lim_{y\to\rho_c^-} H(y) = -\infty$ *and hence* $\lim_{y\to\rho_c^-} A(y) = 0$. *Moreover,* $A(y) \sim K(\rho_c - y)$ *when* $y \to \rho_c^-$ *where* $0 < K < +\infty$ *and in fact,* $K = \frac{2B(\rho_c)}{C'(\rho_c)}$.
If $\rho_c < \rho_b \leq 1$ *then* $\lim_{y\to\rho_c^-} H(y) = +\infty$.

Proof. Easy.

Lemma 4.2. (i) *For any* $i \geq 1$ *and* $|s| < 1$,

$$\frac{C(s)}{2} \cdot G_i''(s) + B(s) \cdot G_i'(s) = a_i - s^i. \tag{4.4}$$

Moreover, for $|s| < \rho_c$, *we have*

$$G_i'(s) \cdot A(s) - G_i'(0) = \int_0^s \frac{2(a_i - y^i)}{C(y)} \cdot A(y)dy. \tag{4.5}$$

(ii) *For any* $i \geq 1$,

$$\lim_{s\to\zeta_c} G_i'(s)A(s) = 0. \tag{4.6}$$

Proof. Integrating (2.2) with respect to $t \in [0,\infty)$ immediately yields (4.4) and then (4.5) immediately follows.

We now turn to prove (4.6). If $-1 < \zeta_c < 0$ then the proof is easy. Indeed, since for this case we have $|G_i'(\zeta_c)| < \infty$ and thus by Lemma 4.1 we know that $\lim_{s\to\zeta_c} A(s) = 0$ and then (4.6) follows.

If $\zeta_c = -1$ the proof is little bit lengthy. Recall Lemma 4.1, we know that if $\zeta_c = -1$ then $C(-x) = C(x)$ for any $x \in [0,1]$. Note that for any $x \in (0,1)$,

$$B(-x) + B(x) = 2\sum_{k=0}^{\infty} b_{2k}x^{2k} > 2b_0, \quad \text{i.e.,} \quad -B(-x) < B(x) - 2b_0.$$

Therefore, for any $s \in [0,1)$,

$$A(-s) = e^{\int_0^{-s}\frac{2B(x)}{C(x)}dx} = e^{-\int_0^{s}\frac{2B(-x)}{C(x)}dx} \le e^{\int_0^{s}\frac{2B(x)-4b_0}{C(x)}dx} = A(s)e^{-4b_0\int_0^s\frac{dx}{C(x)}}.$$

It follows from (4.5) that for $s \in [0,1)$,

$$\begin{aligned}
|G_i'(-s)|A(-s) &\le G_i'(s)A(s)\cdot e^{-4b_0\int_0^s\frac{dx}{C(x)}} \\
&\le G_i'(0)\cdot e^{-4b_0\int_0^s\frac{dx}{C(x)}} + \left(2\int_0^s \frac{A(y)}{C(y)}dy\right)\cdot e^{-4b_0\int_0^s\frac{dx}{C(x)}} \\
&\le G_i'(0)\cdot e^{-4b_0\int_0^s\frac{dx}{C(x)}} + 2\left(\int_0^s \frac{e^{2b_0\int_0^y\frac{dx}{C(x)}}}{C(y)}dy\right)\cdot e^{-4b_0\int_0^s\frac{dx}{C(x)}} \\
&\le G_i'(0)\cdot e^{-4b_0\int_0^s\frac{dx}{C(x)}} + 2\left(\int_0^s \frac{e^{-b_0\int_0^y\frac{dx}{C(x)}}}{C(y)}dy\right)\cdot e^{-b_0\int_0^s\frac{dx}{C(x)}} \\
&\le G_i'(0)\cdot e^{-4b_0\int_0^s\frac{dx}{C(x)}} + \frac{2}{b_0}\cdot e^{-b_0\int_0^s\frac{dx}{C(x)}}.
\end{aligned}$$

Therefore

$$\lim_{s\to\zeta_c}|G_i'(s)|A(s) = \lim_{s\to 1}|G_i'(-s)|A(-s) = 0.$$

and thus (4.6) is proved. □

We are now ready to consider the case that Q is regular. The main conclusions regarding extinction probability for this regular case will be provided below. The irregular, i.e. explosive case will be discussed in the next section. First recall that by Theorem 3.1 an IBCP q−matrix Q is regular if and only if $C'(1) \le 0$.

Theorem 4.1. *Suppose that $C'(1) \le 0$ and $B'(1) \le 0$. Then $a_i = 1$ $(i \ge 1)$.*

Proof. Since $C'(1) \le 0$ and $B'(1) \le 0$, we know by Lemma 2.1 that $C(s) > 0$ and $B(s) > 0$ for all $s \in [0,1)$. It follows from (4.4) that $a_i - s^i \ge 0$ for all $s \in [0,1)$ and thus by letting $s \to 1$ we obtain $a_i \ge 1$. But $a_i \le 1$ is always true and thus $a_i = 1$. □

Theorem 4.2. *Suppose that $C'(1) \le 0$ and $0 < B'(1) \le +\infty$. Then $a_i = 1$ $(i \ge 1)$ if and only if $J = +\infty$ where*

$$J = \int_{\zeta_c}^{1} \frac{A(y)}{C(y)} dy \tag{4.7}$$

and $A(y)$ is defined in (4.3)*. Moreover, if $J < +\infty$ then*

$$a_i = J^{-1} \cdot \int_{\zeta_c}^{1} \frac{y^i A(y)}{C(y)} dy, \quad i \ge 1. \tag{4.8}$$

Proof. Suppose that $J = +\infty$. By the communicating property of the positive states, we know that either $a_i = 1$ for all $i \ge 1$ or $a_i < 1$ for all $i \ge 1$. Now, assume that $a_1 < 1$. We rewrite (4.5) as

$$G_1'(s) \cdot A(s) - G_1'(0) = 2\int_0^{a_1} \frac{a_1 - y}{C(y)} \cdot A(y)dy + 2\int_{a_1}^{s} \frac{a_1 - y}{C(y)} \cdot A(y)dy. \tag{4.9}$$

Let $s \to 1$ in the above equality (4.9). Then the first term on the right hand side of (4.9) is obviously a finite constant and the last term tends to $-\infty$ since $J = +\infty$. The above latter fact could be easily seen by applying the integral mean-values theorem. However, the left hand side is either finite or $+\infty$. This contradiction yields the result $a_i = 1$ for all $i \ge 1$.

Now suppose that $J < +\infty$. By (4.5) and (4.6),

$$-G_i'(0) = -2\int_{\zeta_c}^{0} \frac{a_i - y^i}{C(y)} \cdot A(y)dy. \tag{4.10}$$

It follows from (4.10) and (4.5) that for $i \ge 1$ and $s \in (\zeta_c, 1)$,

$$G_i'(s) \cdot A(s) = \int_{\zeta_c}^{s} \frac{2(a_i - y^i)}{C(y)} \cdot A(y)dy. \tag{4.11}$$

Define $x_i = J^{-1} \cdot \int_{\zeta_c}^{1} \frac{y^i A(y)}{C(y)} dy$ $(i \ge 1)$. Then by (4.11), $a_i \ge x_i$ $(i \ge 1)$. On the other hand, it can be shown that $(x_i; i \ge 1)$ is a solution of the equation

$$\sum_{k=1}^{\infty} q_{ik} x_k + q_{i0} = 0, \quad 0 \le x_i \le 1, i \ge 1.$$

Indeed, for $i \ge 1$,

$$\begin{aligned} & \sum_{k=1}^{\infty} q_{ik} x_k + q_{i0} \\ = & \frac{1}{2J}[i(i-1)\int_{\zeta_c}^{1} y^{i-2}A(y)dy + i\int_{\zeta_c}^{1} y^{i-1}A'(y)dy] \\ = & \frac{i}{2J}[A(1) - \zeta^{i-1}A(\zeta_c)] \\ = & 0 \end{aligned}$$

where the last step follows from Lemma 4.1 and the fact that the assumption $J < \infty$ implies $A(1) = 0$. Therefore, by Lemma 4.46 of Chen [7] (or Li and Chen [12]), we know that $a_i \le x_i$ $(i \ge 1)$. Hence (4.8) is proved. The proof is complete. □

Remark 4.1. It is easy to see that $J < +\infty$ if and only if $J_0 = \int_0^1 \frac{A(y)}{C(y)} dy < +\infty$ where the former J is given in (4.7). Note that the latter one is much easier to be checked since we do not need to know the value of ζ_c which may be quite difficult to obtain.

In practice, it may be difficult to check whether J or even J_0 is finite or not. Fortunately, in most practical cases, we do not need to check either of them. The following corollaries give some convenient sufficient conditions.

Corollary 4.1. *Suppose that $C'(1) < 0$ and $0 < B'(1) < +\infty$. Then $a_i = 1$ $(i \ge 1)$.*

Proof. Note that $A(1) = e^{\int_0^1 \frac{2B(x)}{C(x)} dx} > 0$ and thus $J = +\infty$, the required conclusion immediately follows from Theorem 4.2. □

If $C'(1) < 0$ but $B'(1) = +\infty$, then to guarantee that $a_i = 1$ we need some further conditions. The next corollary provides such a simple sufficient condition which could be easily applied.

Corollary 4.2. *Suppose $C'(1) < 0$ and $B'(1) = +\infty$. Further assume that there exists a $\delta > 0$ and some positive constant k such that*

$$B(s) \sim (-k)(1-s)^{\delta} \qquad as \quad s \uparrow 1$$

then for all $i \ge 1$, we have $a_i = 1$.

Proof. Just note that we have $\frac{B(s)}{C(s)} \sim \frac{(-1)k_1}{(1-s)^{1-\delta}}$ $(s \uparrow 1)$ where $0 < k_1 < \infty$ under the condition of this corollary. Hence $\int_0^1 \frac{B(x)}{C(x)} dx$ is convergent that guarantees that $A(y)$ is bounded on $[0,1]$ which, in turn, implies that $J_0 = +\infty$. □

Of course, the only possible value of $\delta > 0$ in Corollary 4.2 is that $\delta < 1$ since $B'(1) = +\infty$ implies that $\lim_{s\uparrow 1} \frac{(-1)B(s)}{1-s} = +\infty$. Now we turn to consider the case that $C'(1) = 0$.

Corollary 4.3. *Suppose that $C'(1) = 0$ and $0 < B'(1) < +\infty$ together with the further assumptions that both $C''(1)$ and $B''(1)$ are finite. If $C''(1) \ge 4B'(1)$ then $J = +\infty$ and thus $a_i = 1$ $(i \ge 1)$, while if $C''(1) < 4B'(1)$ then $J < +\infty$ and thus $a_i < 1$ $(i \ge 1)$ and then a_i is given by (4.8).*

Proof. Suppose that $C'(1) = 0$. Denote $\gamma = \frac{4B'(1)}{C''(1)}$. First note we have the obvious fact that $C''(1) > 0$ and thus $0 < \gamma < +\infty$ under the condition of this corollary. Secondly, let $g(x) = \frac{2(1-x)B(x)}{C(x)}$, then $g(x)$ is an analytic function on the disk $\{z; |z| < |\zeta_c|\}$ where ζ_c is the unique negative zero of $C(x)$ (also notice that $C(x)$ has no zero in $[0,1)$) and thus could be expanded as a power series of x in the internal $[0,1)$ with the form $g(x) = \sum_{k=0}^{\infty} g_k x^k$, say. Then the $H(y)$ defined in (4.2) for $y \ge 0$ could be expressed as

$$H(y) = \frac{1}{2} \sum_{k=0}^{\infty} g_k \int_0^y \frac{x^k}{1-x} dx$$

which could be rewritten as, by simply letting $x = 1 - (1-x)$,

$$H(y) = -\frac{\ln(1-y)}{2}\sum_{k=0}^{\infty} g_k + H_1(y)$$

where $H_1(y) = \frac{1}{2}\sum_{k=1}^{\infty} g_k \int_0^y \sum_{m=1}^{k}(-1)^m(1-x)^{m-1}dx$ (a further simple form for $H_1(y)$ is of course available but not necessary here) is a bounded function of y on $[0,1]$.

Now noting that

$$\sum_{k=0}^{\infty} g_k = \lim_{x\uparrow 1}\frac{2(1-x)B(x)}{C(x)} = \frac{(-4)B'(1)}{C''(1)} = -\gamma$$

(here we have used the fact that $B''(1) < +\infty$) and therefore the $A(y)$ defined in (4.3) could be written as

$$A(y) = A_1(y)(1-y)^{\gamma}$$

where $A_1(y)$ is a bounded function of $y \in [0,1]$.

It then follows that J_0 (and thus J) is finite if and only if the integral $\int_0^1 \frac{dy}{(1-y)^{2-\gamma}}$ is convergent, or, equivalently, if and only if $4B'(1) > C''(1)$. The proof is now complete. $\square$

Corollary 4.4. *Suppose that $C'(1) = 0$, $0 < C''(1) < 4B'(1) < +\infty$ and $B''(1)$ is finite. Then $a_i \to 0$ as $i \to \infty$ and that*

$$\sum_{i=1}^{\infty} a_i = J^{-1}\int_{\zeta_c}^{1}\frac{yA(y)}{(1-y)C(y)}dy \tag{4.12}$$

which is finite if and only if $\int_0^1 \frac{yA(y)}{(1-y)C(y)}dy < \infty$. More exactly, we have that $a_i \sim \frac{k}{i^{\gamma-1}}$ as $i \to +\infty$ where k is a constant and $\gamma = \frac{4B'(1)}{C''(1)} > 1$. Moreover, $\sum_{i=1}^{\infty} a_i$ is finite if and only if $2B'(1) > C''(1)$.

Proof. By Corollary 4.3 we know that $J < +\infty$ and that for all $i \geq 1$ we have $a_i < 1$ and thus (4.12) follows from (4.8) directly. By a similar approach applied in Corollary 4.3, we could easily show that $a_i \sim k_1 B(i+1, \gamma-1)$ as $i \to +\infty$ where k_1 is a positive constant, $\gamma = \frac{4B'(1)}{C''(1)} > 1$ and $B(a,b)$ is the beta function. Now using the well-known result that $\lim_{z\to+\infty}\frac{\Gamma(z+\alpha)}{\Gamma(z)}z^{-\alpha} = 1$ (which is true even for a complex number α with positive real part) where $\Gamma(x)$ is the gamma function, we immediately obtained the result that there exists some positive constant k such that $a_i \sim \frac{k}{i^{\gamma-1}}$ as $i \to \infty$. Now $a_i \to 0$ also immediately follows.

Furthermore, again, similarly as in Corollary 4.3 we could show that $\sum_{i=1}^{\infty} a_i$ is finite if and only if the integral $\int_0^1 \frac{dy}{(1-y)^{3-\gamma}}$ is convergent or, equivalently, if and only if $2B'(1) > C''(1)$. This finishes the proof. $\square$

Finally, we point out the following easy corollary. If the process takes the initial distribution $\Pi = (\pi_i; i \geq 0)$ rather than a single point distribution, then the extinction probability a_i will be given as below, here the trivial case of $\pi_0 = 1$ will be excluded.

Corollary 4.5. *Suppose that $C'(1) \leq 0$ and the process starts with the initial distribution $\Pi = (\pi_i; i \geq 0)$. Then the extinction probability a_Π is given by*

(i) *If $B'(1) \leq 0$, then $a_\Pi = 1$.*

(ii) *If $0 < B'(1) \leq +\infty$ and $J_0 = \int_0^1 \frac{A(y)}{C(y)} = +\infty$, then $a_\Pi = 1$.*

(iii) *If $0 < B'(1) \leq +\infty$ and $J_0 = \int_0^1 \frac{A(y)}{C(y)} < +\infty$, then*

$$a_\Pi = J^{-1} \cdot \int_{\zeta_c}^1 \frac{A(y)}{\Pi(y)C(y)} dy$$

where $\Pi(y) = \sum_{k=0}^\infty \pi_k y^k$.

In the final part of this section, we consider the mean extinction time for the regular case.

Theorem 4.3. *Suppose that Q is given in (1.1) – (1.2) and $P(t) = (p_{ij}(t); i, j \geq 0)$ is the Feller minimal Q-function. If $C'(1) \leq 0$ and $J = +\infty$, then $E_i[\tau_0] < \infty$ $(i \geq 1)$ if and only if*

$$\int_0^1 [\frac{1}{A(s)} \int_0^s \frac{(1-y)A(y)}{C(y)} dy] ds < \infty \tag{4.13}$$

and $E_i[\tau_0]$ is given by

$$E_i[\tau_0] = \int_0^1 [\frac{1}{A(s)} \int_{\zeta_c}^s \frac{2(1-y^i)A(y)}{C(y)} dy] ds, \quad i \geq 1 \tag{4.14}$$

where E_i is the conditional expectation when the process starts at $i \geq 1$.

Proof. By (4.5) and noting $\zeta_c > -1$ we obtain

$$G_i'(s)A(s) - G_i'(0) = \int_0^s \frac{2(1-y^i)}{C(y)} \cdot A(y) dy \tag{4.15}$$

and

$$G_i'(0) = \int_{\zeta_c}^0 \frac{2(1-y^i)}{C(y)} \cdot A(y) dy. \tag{4.16}$$

Integrating (4.15) and noting $G_i(0) = 0$ yield that

$$G_i(s) = G_i'(0) \int_0^s \frac{du}{A(u)} + \int_0^s [\frac{1}{A(u)} \cdot \int_0^u \frac{2(1-y^i)}{C(y)} \cdot A(y) dy] du.$$

By substituting (4.16) into the above equality we immediately obtain

$$G_i(s) = \int_0^s [\frac{1}{A(u)} \cdot \int_{\zeta_c}^u \frac{2(1-y^i)}{C(y)} \cdot A(y) dy] du. \tag{4.17}$$

Letting $s \to 1$ in (4.17) yields that

$$G_i(1) = \int_0^1 [\frac{1}{A(u)} \cdot \int_{\zeta_c}^u \frac{2(1-y^i)}{C(y)} \cdot A(y) dy] du. \tag{4.18}$$

By noting that $G_i(1) = \sum_{j=1}^{\infty} \int_0^{\infty} p_{ij}(t)dt = \int_0^{\infty}(1-p_{i0}(t))dt$ we immediately see that (4.14) follows from (4.18) directly. It is also obvious that (4.14) is finite if and only if (4.13) is true. □

If $C'(1) \leq 0$ and $J < +\infty$, then, trivially, $E_i[\tau_0] = +\infty$ due to the fact that $a_i < 1$. To counteract this, we turn to consider the informative conditional mean extinction time. Similarly as above, we may obtain the following conclusion.

Theorem 4.4. *Suppose that Q is an IBC q-matrix as in (1.1)-(1.2) and $P(t) = (p_{ij}(t); i, j \geq 0)$ is the unique IBCP, i.e. the Feller minimal Q-function. If $C'(1) \leq 0$ and $J < +\infty$, then*

$$a_i E_i[\tau_0|\tau_0 < \infty] = \int_0^1 [\frac{1}{A(s)} \int_{\zeta_c}^s \frac{2(a_i - y^i)A(y)}{C(y)} dy] ds, \quad i \geq 1.$$

5. Extinction Probability: Irregular case

In the previous section, we have considered the situation that Q is regular and satisfactory results regarding extinction properties have been obtained. We now turn to consider the case when Q is irregular, or explosive. That is that we now consider the case that $0 < C'(1) < +\infty$ and thus $\rho_c < 1$. Note that in this case the unique IBCP is still the Feller minimal Q-process and thus dishonest. We shall see that this irregular case is much more subtle and it is necessary to further divide it into a few sub-categories.

Definition 5.1. An irregular IBC q-matrix Q is called super-explosive, critical-explosive or sub-explosive, if $\rho_b < \rho_c, \rho_b = \rho_c$ or $\rho_b > \rho_c$, respectively, where $\rho_c < 1$ and $\rho_b \leq 1$ are the smallest nonnegative zeros of C(s) and $B(s)$, respectively.

As extinction properties are concerned, the critical-explosive is the most simple case. Indeed, we have the following simple conclusion.

Theorem 5.1. *If Q is critical-explosive, i.e., $\rho_c = \rho_b < 1$, then $a_i = \rho_c^i$ $(i \geq 1)$.*

Proof. Note the facts that $\rho_c < 1$ is the common zero of $B(s)$ and $C(s)$ and both $G_i'(\rho_c)$ and $G_i''(\rho_c)$ are finite, thus by letting $s = \rho_c$ in (4.4) we immediately get the conclusion. □

Under the condition that Q takes the form when the corresponding $B(s)$ and $C(s)$ are polynomials of degree 2 and 3 respectively, Theorem 5.1 was proved in Kalinkin [10]. One can see that by our Theorem 5.1, this conclusion holds true for all situations. For more details, see our later Example 6.1.

For this critical-explosive case, Kalinkin [10] didn't consider the conditional mean extinction time. Here we would like to provide such conclusions for general case.

Theorem 5.2. *If Q is critical-explosive, then the mean conditional extinction time $E_i[\tau_0|\tau_0 < \infty]$ is given by*

$$E_i[\tau_0|\tau_0 < \infty] = \rho_c^{-i} \int_0^{\rho_c} [\frac{2}{A(s)} \int_{\zeta_c}^s (1 - (\frac{y}{\rho_c})^i) \frac{A(y)}{C(y)} dy] ds$$

where $A(s)$ is defined in (4.3) and its domain is $(\zeta_c, 1)$.

Proof. First note that $\rho_c < 1$ is the unique common simple zero of $B(s)$ and $C(s)$ on $[0, 1)$ and thus $H(s)$ and $A(s)$ given in (4.2) and (4.3) are all well-defined on $[0, 1)$. Hence, by solving

$$\frac{C(s)}{2}G_i''(s) + B(s)G_i'(s) = \rho_c^i - s^i,$$

we can prove this theorem by a similar argument as in Theorem 4.3. □

We now turn to the super-explosive case. Interestingly, a closed form of extinction probability is still available for this case.

Theorem 5.3. *Suppose that $\rho_b < \rho_c < 1$. Then for any $i \geq 1$, the extinction probability a_i of the IBCP starting from $i \geq 1$, is given by*

$$a_i = \frac{\int_{\zeta_c}^{\rho_c} \frac{y^i A(y)}{C(y)} dy}{\int_{\zeta_c}^{\rho_c} \frac{A(y)}{C(y)} dy}. \tag{5.1}$$

Proof. Integrating (2.2) with respect to $t \in [0, \infty)$ yields that

$$\frac{C(s)}{2}G_i''(s) + B(s)G_i'(s) = a_i - s^i, \quad i \geq 1, |s| < 1. \tag{5.2}$$

Solving (5.2) on $[0, \rho_c)$ yields that

$$G_i'(s) \cdot e^{\int_0^s \frac{2B(x)}{C(x)}dx} - G_i'(0) = \int_0^s \frac{2(a_i - y^i)}{C(y)} \cdot e^{\int_0^y \frac{2B(x)}{C(x)}dx} dy.$$

Noting $G_i'(\rho_c) < +\infty$ and $\int_0^{\rho_c} \frac{2B(x)}{C(x)}dx = -\infty$ (since $B(x) < 0$ for $x \in (\rho_b, \rho_c)$) we obtain

$$-G_i'(0) = \int_0^{\rho_c} \frac{2(a_i - y^i)}{C(y)} \cdot e^{\int_0^y \frac{2B(x)}{C(x)}dx} dy. \tag{5.3}$$

Similarly, solving (5.2) on $(\zeta_c, 0]$ yields that

$$G_i'(s) \cdot e^{\int_0^s \frac{2B(x)}{C(x)}dx} - G_i'(0) = \int_0^s \frac{2(a_i - y^i)}{C(y)} \cdot e^{\int_0^y \frac{2B(x)}{C(x)}dx} dy.$$

Noting $G_i'(\zeta_c) < +\infty$ and $\int_0^{\zeta_c} \frac{2B(x)}{C(x)}dx = -\infty$ (since $B(x) > 0$ for $x \in (\zeta_c, 0)$) we obtain

$$-G_i'(0) = \int_0^{\zeta_c} \frac{2(a_i - y^i)}{C(y)} \cdot e^{\int_0^y \frac{2B(x)}{C(x)}dx} dy. \tag{5.4}$$

It follows from (5.3) and (5.4) that

$$\int_{\zeta_c}^{\rho_c} \frac{2(a_i - y^i)}{C(y)} \cdot e^{\int_0^y \frac{2B(x)}{C(x)}dx} dy = 0$$

and thus (5.1) follows. □

Again, Kalinkin [10] obtained this theorem under the condition that Q takes the special form as in our later Example 6.1.

Remark 5.1. By formula (5.1), we could see that we have the following asymptotic behaviour for the extinction probability a_i, i.e.

$$a_i \sim k i^{\alpha} \rho_c^i \qquad \text{as } i \to +\infty \tag{5.5}$$

where k and α are two positive constants. Indeed, it is not difficult to show that the extinction probability could be rewritten as $a_i = kB(i+1, \alpha)$ for two positive constants k and α where $B(a, b)$ is the beta function. Using the well-known result $\lim_{z\to+\infty} \frac{\Gamma(z+\alpha)}{\Gamma(z)} z^{-\alpha} = 1$, once again, where $\Gamma(x)$ is the gamma function, we immediately obtain (5.5). A further detailed investigation could show that the constant α is just $\frac{2B(\rho_c)}{C'(\rho_c)}$ which is positive since both $B(\rho_c)$ and $C'(\rho_c)$ are negative. Hence we actually have

$$a_i \sim k i^{\frac{2B(\rho_c)}{C'(\rho_c)}} \rho_c^i \qquad \text{as } i \to +\infty. \tag{5.6}$$

This interesting fact has been already observed by Kalinkin [10].

We next consider the sub-explosive case. Surprisingly, this case is very subtle. As we are aware, for this subtle case, few results have been obtained until now. Even for the very special case as in our Example 6.1, this is still an open problem. For example, this case was not considered in Kalinkin [10]. This seems to indicate that the sub-explosive case is indeed challenging. Before we formally tackle this question, we first show that a trivial lower bound as well as upper bound could be easily given.

Theorem 5.4. *Suppose that the IBC q-matrix Q is sub-explosive, i.e., $\rho_c < \rho_b \leq 1$, then $\rho_c^i < a_i < \rho_b^i$ $(i \geq 1)$.*

Proof. Letting $s = \rho_c$ and ρ_b in (2.5) and using the dominated convergence theorem together with Lemma 2.1 immediately yields the result. Note that, however, if $\rho_c < \rho_b = 1$, then the fact that $a_i < \rho_b^i = 1$ follows from the fact that the IBCP is not honest, see Theorem 3.1. □

But we could do much better than Theorem 5.4. To this end, we need to define a simple transformation as follows.

Firstly we rewrite the basic equation (4.4) as

$$A_0(s)G_i''(s) + B_0(s)G_i'(s) = U_0(s) \tag{5.7}$$

where $A_0(s) = \frac{C(s)}{2}$, $B_0(s) = B(s)$ and $U_0(s) = a_i - s^i$. Now if we define

$$A_1(s) = A_0(s)B_0(s)$$

$$B_1(s) = B_0(s)[B_0(s) + A_0'(s)] - A_0(s)B_0'(s)$$

$$U_1(s) = B_0(s)U_0'(s) - B_0'(s)U_0(s)$$

then we get

$$A_1(s)G_i'''(s) + B_1(s)G_i''(s) = U_1(s).$$

Recursively, we could easily get that for any $n \geq 0$,

$$A_n(s)G_i^{(n+2)}(s) + B_n(s)G_i^{(n+1)}(s) = U_n(s) \tag{5.8}$$

where $\{A_n(s), B_n(s), U_n(s)\}$ are defined recursively as

$$A_n(s) = A_{n-1}(s)B_{n-1}(s) \tag{5.9}$$

$$B_n(s) = B_{n-1}(s)[B_{n-1}(s) + A'_{n-1}(s)] - A_{n-1}(s)B'_{n-1}(s) \tag{5.10}$$

$$U_n(s) = B_{n-1}(s)U'_{n-1}(s) - B'_{n-1}(s)U_{n-1}(s). \tag{5.11}$$

Note that all $A_n(s)$ and $B_n(s)$ ($n \geq 0$) are entirely expressible in terms of the given functions $B(s)$ and $C(s)$ and also independent of $i \geq 1$. Similarly, all $U_n(s)$ are totally expressible in terms of $B(s)$ and $C(s)$ together with the unknown constant a_i. In particular, all $\{A_n(s), B_n(s), U_n(s)\}$ are power series of s and thus analytic on the complex disk $\{z; |z| < R\}$ where $R = R_b \wedge R_c$ and R_b and R_c are convergence radius of the two power series $B(s)$ and $C(s)$ respectively. Of course, we have $R \geq 1$.

By (5.9) we immediately obtain that

$$A_n(s) = A_0(s) \prod_{k=0}^{n-1} B_k(s). \tag{5.12}$$

Noting that $A_0(s) = \frac{C(s)}{2}$ immediately yields that

$$A_n(\rho_c) = A_n(\zeta_c) = 0, \quad \forall n \geq 0.$$

It then easily follows that

$$B_n(\rho_c) = (B_0(\rho_c) + nA'_0(\rho_c)) \prod_{k=0}^{n-1} B_k(\rho_c), \qquad \forall n \geq 1 \tag{5.13}$$

$$A'_n(\rho_c) = A'_0(\rho_c)) \prod_{k=0}^{n-1} B_k(\rho_c), \qquad \forall n \geq 1 \tag{5.14}$$

with

$$B_0(\rho_c) = B(\rho_c) > 0 \ \text{ and } \ A'_0(\rho_c) = \frac{C'(\rho_c)}{2} < 0. \tag{5.15}$$

Similarly,

$$B_n(\zeta_c) = (B_0(\zeta_c) + nA'_0(\zeta_c)) \prod_{k=0}^{n-1} B_k(\zeta_c), \quad \forall n \geq 1 \tag{5.16}$$

$$A'_n(\zeta_c) = A'_0(\zeta_c)) \prod_{k=0}^{n-1} B_k(\zeta_c), \quad \forall n \geq 1 \tag{5.17}$$

with

$$B_0(\zeta_c) = B(\zeta_c) > 0 \text{ and } A'_0(\zeta_c) = \frac{C'(\zeta_c)}{2} > 0. \tag{5.18}$$

Regarding the signs of $B_n(\rho_c)$ and $B_n(\zeta_c)$ which are important in the later discussion, we have the following simple conclusions.

Lemma 5.1. *Suppose Q is sub-explosive. Then we have*
(i) $A_n(\rho_c) = A_n(\zeta_c) = 0, \quad \forall n \geq 0$;
(ii) $B_n(\zeta_c) > 0, \quad \forall n \geq 0$;
(iii) *If $-\frac{2B(\rho_c)}{C'(\rho_c)}$ is a positive integer m, say, then $B_n(\rho_c) > 0$ for all $0 \leq n \leq m-1$ and $B_m(\rho_c) = 0$. If $-\frac{2B(\rho_c)}{C'(\rho_c)}$ is not a positive integer, then $B_n(\rho_c) > 0$ for all $0 \leq n < m$ and $B_m(\rho_c) < 0$ where $m = [\frac{-2B(\rho_c)}{C'(\rho_c)}]$ is the integer part of $\frac{-2B(\rho_c)}{C'(\rho_c)}$.*

Proof. (i) has been shown before. Since $B_0(\zeta_c) > 0$ and $A'_0(\zeta_c) > 0$, one can easily get (ii) by the mathematical induction. Finally, (iii) follows directly from (5.13). □

We are now ready to provide more information about the extinction properties for the sub-explosive case, i.e., $\rho_c < \rho_b \leq 1$. We now first show that we could do much better than Theorem 5.1 by providing much narrower bounds.

Theorem 5.5. *Suppose that $\rho_c < \rho_b \leq 1$.*
(i) *If $C'(\rho_c) + 2B(\rho_c) = 0$, then*

$$a_i = \rho_c^i + i\sigma\rho_c^{i-1} \tag{5.19}$$

where the positive constant σ is independent of i and given by

$$\sigma = -\frac{B(\rho_c)}{B'(\rho_c)}.$$

(ii) *If $C'(\rho_c) + 2B(\rho_c) > 0$, then*

$$\rho_c^i + i\sigma\rho_c^{i-1} < a_i < \rho_b^i \tag{5.20}$$

where σ is the same as in (5.19).
(iii) *If $C'(\rho_c) + 2B(\rho_c) < 0$, then*

$$\rho_c^i < a_i < \min\{\rho_b^i, \rho_c^i + i\sigma\rho_c^{i-1}\}. \tag{5.21}$$

In particular, if $\rho_b = 1$, then

$$\rho_c^i < a_i < \rho_c^i + i\sigma\rho_c^{i-1}. \tag{5.22}$$

Proof. By letting $n = 1$ in (5.8)-(5.11) together with the facts $B_0(s) = B(s), C_0(s) = \frac{C(s)}{2}$ and $U_0(s) = a_i - s^i$, we see that

$$\begin{aligned} & \frac{C(s)B(s)}{2} G_i'''(s) + \frac{B(s)(2B(s) + C'(s)) - C(s)B'(s)}{2} G_i''(s) \\ = \; & B'(s)(s^i - a_i) - is^{i-1}B(s) \end{aligned} \tag{5.23}$$

Letting $s = \rho_c$ in (5.23) and noting $C(\rho_c) = 0$ together with the facts that $G_i'''(\rho_c) < \infty$ and $G_i''(\rho_c) < \infty$ yields that

$$[B(\rho_c)(2B(\rho_c) + C'(\rho_c))]G_i''(\rho_c) = 2B'(\rho_c)(\rho_c^i - a_i) - 2i\rho_c^{i-1}B(\rho_c) \tag{5.24}$$

Now, noting $0 < G_i''(\rho_c) < \infty$ and using some easy algebra we obtain all the conclusions. □

By Theorem 5.5 we see that if $C'(\rho_c) + 2B(\rho_c) = 0$, then the exact values of a_i $(i \geq 1)$ are given, while if $C'(\rho_c) + 2B(\rho_c) < 0$, only better bounds are provided. In fact , for the latter case, an explicit expression for a_i $(i \geq 1)$ is available.

Theorem 5.6. *Suppose the IBC q-matrix Q is sub-explosive and $C'(\rho_c) + 2B(\rho_c) < 0$, then we have*

$$a_i = \frac{\int_{\zeta_c}^{\rho_c} \frac{y^i B'(y) - iy^{i-1}B(y)}{A_1(y)} e^{\int_0^y \frac{B_1(x)}{A_1(x)}dx} dy}{\int_{\zeta_c}^{\rho_c} \frac{B'(y)}{A_1(y)} e^{\int_0^y \frac{B_1(x)}{A_1(x)}dx} dy}.$$

Proof. By (5.24), we know that for $|s| < \rho_c$,

$$A_1(s)G_i'''(s) + B_1(s)G_i''(s) = U_1(s) \tag{5.25}$$

where

$$A_1(s) = \frac{C(s)B(s)}{2}, \quad B_1(s) = \frac{B(s)(2B(s) + C'(s) - C(s)B'(s))}{2}$$

and

$$U_1(s) = B'(s)(s^i - a_i) - is^{i-1}B(s).$$

Noting that $A_1(s) > 0$ for all $s \in (\zeta_c, \rho_c)$ and $G''(\zeta_c)e^{\int_0^{\zeta_c} \frac{B_1(y)}{A_1(y)}dy} = 0$, we can solve (5.25) to get

$$G''(s)e^{\int_0^s \frac{B_1(y)}{A_1(y)}dy} = \int_{\zeta_c}^s \frac{U_1(y)e^{\int_0^y \frac{B_1(x)}{A_1(x)}dx}}{A_1(y)} dy. \tag{5.26}$$

Since $B_1(\rho_c) = \frac{1}{2}B(\rho_c)(C'(\rho_c) + 2B(\rho_c)) < 0$, we have $G''(\rho_c)e^{\int_0^{\rho_c} \frac{B_1(y)}{A_1(y)}dy} = 0$. Therefore, letting $s \uparrow \rho_c$ in (5.26) yields that

$$\int_{\zeta_c}^{\rho_c} \frac{U_1(y)e^{\int_0^y \frac{B_1(x)}{A_1(x)}dx}}{A_1(y)} dy = 0.$$

i.e.,

$$a_i \int_{\zeta_c}^{\rho_c} \frac{B'(y)}{A_1(y)} e^{\int_0^y \frac{B_1(x)}{A_1(x)}dx} dy = \int_{\zeta_c}^{\rho_c} \frac{y^i B'(y) - iy^{i-1}B(y)}{A_1(y)} e^{\int_0^y \frac{B_1(x)}{A_1(x)}dx} dy. \tag{5.27}$$

Note that $B'(y) \leq B'(|y|) \leq B'(\rho_c) < 0$ for all $s \in (\zeta_c, \rho_c)$, it follows that

$$a_i = \frac{\int_{\zeta_c}^{\rho_c} \frac{y^i B'(y) - iy^{i-1}B(y)}{A_1(y)} e^{\int_0^y \frac{B_1(x)}{A_1(x)}dx} dy}{\int_{\zeta_c}^{\rho_c} \frac{B'(y)}{A_1(y)} e^{\int_0^y \frac{B_1(x)}{A_1(x)}dx} dy}.$$

The proof is complete. □

By Theorems 5.5 and 5.6, we see that as the closed form of extinction probability is concerned we only need to investigate the case of $C'(\rho_c) + 2B(\rho_c) > 0$. In order to further examine this even subtle case, we need a little more algebra.

Firstly, we need to know more structure of $U_n(s)$ defined in (5.11). Note that $U_n(s)$ depends upon $i \geq 1$. To emphasize this, we shall denote it as $U_{ni}(s)$.

Lemma 5.2. *For any $n \geq 1$ and $i \geq 1$, we have*

$$U_{ni}(s) = \sum_{k=0}^{n} D_{n,k}(s) U_{0i}^{(k)}(s) \tag{5.28}$$

where $U_{0i}^{(k)}(s)$ denotes the k'th derivative of $U_{0i}(s) = a_i - s^i$, $\{D_{n,k}(s), 0 \leq k \leq n\}$ are totally expressible by the two known functions $B(s)$ and $C(s)$. In particular, $D_{n,k}(s)$ does not depends on $i \geq 1$. Moreover, $\{D_{n,k}(s)\}$ can be given recursively as follows

$$D_{1,0}(s) = -B'(s), \qquad D_{1,1}(s) = B(s) \tag{5.29}$$

$$D_{n,k}(s) = D_{n-1,k-1}(s)B_{n-1}(s) - D_{n-1,k}(s)B'_{n-1}(s) + D'_{n-1,k}(s)B_{n-1}(s), \quad k \leq n-1 \tag{5.30}$$

and

$$D_{n,n}(s) = \prod_{m=0}^{n-1} B_m(s). \tag{5.31}$$

Proof. Applying induction principle in (5.11) immediately yields all the conclusions. □

Remark 5.2. By the definition of $U_{0i}(s)$ we see that $U_{ni}(s) = \sum_{k=0}^{n \wedge i} D_{n,k}(s) U_{0i}^{(k)}(s)$. However, for notational convenience, we shall not do so.

We now further consider the case that $C'(\rho_c) + 2B(\rho_c) > 0$. Considering $B(\rho_c) > 0$ and $C'(\rho_c) < 0$, there exists a positive integer m such that

$$mC'(\rho_c) + 2B(\rho_c) \leq 0 \text{ and } (m-1)C'(\rho_c) + 2B(\rho_c) > 0.$$

Our Theorems 5.5 and 5.6 have tackled the case $m = 1$. We now consider the general case for $m \geq 2$.

Theorem 5.7. *Suppose that Q is a sub-explosive IBC-q-matrix with $C'(\rho_c) + 2B(\rho_c) > 0$. Let $m = \min\{k \in \mathbf{Z}_+; kC'(\rho_c) + 2B(\rho_c) \leq 0\}$ and thus $m \geq 2$.*

(i) *If $mC'(\rho_c) + 2B(\rho_c) = 0$, then $U_k(\rho_c) > 0$ for all $0 \leq k \leq m-1$ and $U_m(\rho_c) = 0$. Hence*

$$a_i = \rho_c^i + \sum_{k=1}^{m\wedge i} \frac{D_{m,k}(\rho_c)}{D_{m,0}(\rho_c)} \cdot \frac{i!}{(i-k)!} \cdot \rho_c^{i-k}. \tag{5.32}$$

In particular, $a_1 = \rho_c + \frac{D_{m,1}(\rho_c)}{D_{m,0}(\rho_c)}$.

(ii) *If $mC'(\rho_c) + 2B(\rho_c) < 0$, then $U_k(\rho_c) > 0$ for all $0 \leq k \leq m-1$ and $U_m(\rho_c) < 0$. Hence*

$$\rho_c^i + \sum_{k=1}^{(m-1)\wedge i} \frac{D_{m-1,k}(\rho_c)}{D_{m-1,0}(\rho_c)} \cdot \frac{i!}{(i-k)!} \cdot \rho_c^{i-k} < a_i < \rho_c^i + \sum_{k=1}^{m\wedge i} \frac{D_{m,k}(\rho_c)}{D_{m,0}(\rho_c)} \cdot \frac{i!}{(i-k)!} \cdot \rho_c^{i-k}. \tag{5.33}$$

In particular, $\rho_c + \frac{D_{m-1,1}(\rho_c)}{D_{m-1,0}(\rho_c)} < a_1 < \rho_c + \frac{D_{m,1}(\rho_c)}{D_{m,0}(\rho_c)}$.

Proof. It follows from (5.13) that

$$B_k(\rho_c) = \frac{1}{2}(2B(\rho_c) + kC'(\rho_c)) \prod_{j=0}^{k-1} B_j(\rho_c), \qquad \forall k \geq 1. \tag{5.34}$$

From (5.34) and $B_0(\rho_c) = B(\rho_c) > 0$, one can easily know that

$$m = \min\{k \geq 1; kC'(\rho_c) + 2B(\rho_c) \leq 0\} = \min\{k \geq 1; B_k(\rho_c) \leq 0\}.$$

Recall that we have

$$A_m(s)G_i^{(m+2)}(s) + B_m(s)G_i^{(m+1)}(s) = U_m(s).$$

Letting $s = \rho_c$ in the above equation immediately yields $U_m(\rho_c) = 0$. Then (5.32) immediately follows from (5.34) and Lemma 5.2. Finally, if $mC'(\rho_c)+2B(\rho_c) < 0$, we may similarly prove that $U_k(\rho_c) > 0$ for all $0 \leq k \leq m-1$ and $U_m(\rho_c) < 0$ and hence the inequalities (5.33) follows. □

Remark 5.3. In obtaining (5.32) we have assumed that $D_{m,0}(\rho_c) \neq 0$. Indeed, if $D_{m,0}(\rho_c) = 0$, then by letting $s = \rho_c$ in (5.28) for $i = 1$ (see also Remark 5.2), we would get that $D_{m,1}(\rho_c) = 0$. Now repeating using (5.28) (again, referring to Remark 5.2) for $i = 2, 3, \ldots$ etc, we would get that for all $i \in Z_+$ we have $D_{m,i}(\rho_c) = 0$. This will imply that both $A_m(s)$ and $B_m(s)$ are dividable by $s - \rho_c$ (see (5.30) and (5.31)) and thus the equation $A_m(s)G_i^{(m+2)}(s) + B_m(s)G_i^{(m+1)}(s) = U_m(s)$ could be reduced by dividing $s - \rho_c$. Similarly, in obtaining (5.33) we have further assumed that $D_{m-1,\,0}(\rho_c) > 0$ and $D_{m,0}(\rho_c) > 0$. If they are negative, the inequality (5.33) may need to be reversed. □

Remark 5.4. By (5.19) and (5.32), it is easily seen that if $mC'(\rho_c) + 2B(\rho_c) = 0$, then we still have the following asymptotic behaviour for the extinction probability a_i, that is that

$$a_i \sim k i^{\alpha} \rho_c^i \qquad \text{as } i \to +\infty \tag{5.35}$$

where k is a constant and $\alpha = m = -\frac{2B(\rho_c)}{C'(\rho_c)}$. By comparing this with (5.6) in Remark 5.1 shows that the quantity $\frac{2B(\rho_c)}{C'(\rho_c)}$ is a key quantity in studying the extinction behaviour of the IBCP process. Furthermore, the inequality (5.33) has already provided a strong hint that (5.35) may still hold true with $\alpha = -\frac{2B(\rho_c)}{C'(\rho_c)}$ when $mC'(\rho_c)+2B(\rho_c) < 0$. We shall investigate such interesting behaviour in a subsequent paper.

Although (ii) of Theorem 5.7 yields much better bounds of the extinction probabilities, closed forms are not given. In order to obtain such closed forms, we shall make the following Assumption A. Later we shall see that in some sense, the purpose of this assumption is just to simplify the notations. Recall that we only need to consider the case that $C'(\rho_c)+2B(\rho_c) > 0$, since otherwise, the answer has been already given in Theorem 5.6. We shall, also, assume that $-2B(\rho_c)/C'(\rho_c)$ is not a positive integer since otherwise the problem is also solved, see Theorem 5.7. Now considering $C'(\rho_c) < 0$ and $B(\rho_c) > 0$, we could find the smallest positive integer $k > 1$ such that $kC'(\rho_c) + 2B(\rho_c) < 0$.

Assumption A. For a sub-explosive IBC-q-matrix Q satisfying $C'(\rho_c) + 2B(\rho_c) > 0$, we assume that

$A_m(s) > 0$ for all $s \in (\zeta_c, \rho_c)$ where $m = \min\{k \geq 1; kC'(\rho_c) + 2B(\rho_c) < 0\}$ and $A_m(s)$ is given in (5.10).

We now show that under Assumption A, the closed forms for the extinction probabilities are available.

Theorem 5.8. *Suppose that Q is a sub-explosive IBC-q-matrix satisfying $C'(\rho_c)+2B(\rho_c) > 0$ and that $-2B(\rho_c)/C'(\rho_c)$ is not an integer. Let $m = \min\{k \geq 1; kC'(\rho_c) + 2B(\rho_c) < 0\}$. Further assume that $A_m(s)$ satisfies Assumption A. Then*

$$a_i = \frac{\sum_{k=0}^{m\wedge i} \frac{i!}{(i-k)!} \int_{\zeta_c}^{\rho_c} \frac{y^{i-k} D_{m,k}(y)}{A_m(y)} e^{H_m(y)} dy}{\int_{\zeta_c}^{\rho_c} \frac{D_{m,0}(y)}{A_m(y)} e^{H_m(y)} dy}. \tag{5.36}$$

where $H_m(y) = \int_0^y \frac{B_m(x)}{A_m(x)} dx$ and $D_{m,k}(s)$ etc are given in Lemma 5.2.

Proof. Consider the equation (5.8) with $n = m$, i.e.,

$$A_m(s) G_i^{(m+2)}(s) + B_m(s) G_i^{(m+1)}(s) = U_{mi}(s).$$

Solving the above equation under Assumption A yields that

$$G_i^{(m+1)}(s) e^{H_m(s)} - G_i^{(m+1)}(0) = \int_0^s \frac{U_{mi}(y)}{A_m(y)} e^{H_m(y)} dy. \tag{5.37}$$

By the definition of m and the fact we proved in Theorem 5.7, we know that $B_n(\rho_c) > 0$ for all $0 \leq n \leq m-1$ and $B_m(\rho_c) < 0$. Hence $B_m(s) < 0$ in a sufficient small interval $(\rho_c - \varepsilon, \rho_c)$

(where $\varepsilon > 0$). This, together with the Assumption A, implies $e^{H_m(s)} \to 0$ as $s \to \rho_c^-$ since $A_m(\rho_c) = 0$. Now using the fact that $0 < G_i^{(m+1)}(\rho_c) < +\infty$ and letting $s \to \rho_c^-$ in (5.37) immediately yields

$$-G_i^{(m+1)}(0) = \int_0^{\rho_c} \frac{U_{mi}(y)}{A_m(y)} e^{H_m(y)} dy.$$

On the other hand, by looking at (5.16)-(5.18), we know that $e^{H_m(s)} \to 0$ as $s \to \zeta_c^+$ and thus letting $s \to \zeta_c^+$ in (5.37) and, again, using the fact that $|G_i^{(m+1)}(\xi_c)| < +\infty$ yields that

$$-G_i^{(m+1)}(0) = \int_0^{\zeta_c} \frac{U_{mi}(y)}{A_m(y)} e^{H_m(y)} dy.$$

By combining the above just proven two equalities, we get that

$$\int_{\zeta_c}^{\rho_c} \frac{U_{mi}(y)}{A_m(y)} e^{H_m(y)} dy = 0.$$

Now, substituting (5.28) into the above equality immediately yields (5.36). □

We now turn to consider the situation that the Assumption A fails. Recall that $A_m(s)$ is an analytic function of s and thus has only finitely many zeros in (ζ_c, ρ_c). Without loss of any generality, we may assume that 0 is not a zero of $A_m(s)$, since otherwise we could choose another point which is not a zero of $A_m(s)$ as the starting point of our following discussion. Let μ_0 be the smallest negative zero of $A_m(s)$ on (ζ_c, ρ_c) (if $A_m(s)$ has no negative zero on $(_c, \rho_c)$, we could simply let $\mu_0 = 0$). Now for each negative zero μ_k, say, of $A_m(s)$, we could find a sufficient small radius c_k, such that the $A_m(s)$ has no zero on the disk $\{z; |z - \mu_k| \le c_k\}$ except μ_k itself. Now, for any y such that $\zeta_c < y < \mu_0$ and any analytic function $f(s)$, we could define the complex integral $\oint_{C_y} \frac{f(s)}{A_m(s)} ds$ where C_y is the closed curve starting and ending at y and along each circle $\{z; |z - \mu_k| = c_k\}$ as defined above together with alongside the real number axis (but for $s \le 0$ only). Now this integral is a real value. Indeed, by Theorem of residue in complex analysis, this integral just equals to the sum of residues of the function $\frac{f(s)}{A_m(s)}$ at zeros of $A_m(s)$. By symmetric property, we know that the integral $\int_{\tilde{C}_y} \frac{f(s)}{A_m(s)} ds$ where $\tilde{C}_y$ is just the upper part of C_y, starting from 0 and ending at y, is just half of the above value and thus is again a real value. As a convention, in the following, we shall just simply use $(\sim) \oint_0^y \frac{f(s)}{A_m(s)} ds$ to denote this integral. Of course, this integral does not depend on the above disk $\{z; |z - \mu_k| \le c_k\}$. If necessary, we could use any other curve D_y, say, so long as the closed curve, formed by C_y and D_y, does not contain any zeros of $A_m(s)$. Similarly, let λ_0 be the largest positive zero of $A_m(s)$ on (ζ_c, ρ_c), then for any $\lambda_0 < y < \rho_c$ and any analytic function $f(s)$, the integral $(\sim) \oint_0^y \frac{f(s)}{A_m(s)} ds$ is also well-defined and is real valued by bearing in mind that this integral is not passing any zero of $A_m(s)$. Now by such convention and understanding, we could get the conclusion, taking a similar form as in Theorem 5.8, as follows

Theorem 5.9. *Suppose that Q is a sub-explosive IBC-q-matrix satisfying $C'(\rho_c) + 2B(\rho_c) > 0$ and that $-2B(\rho_c)/C'(\rho_c)$ is not an integer. Let $m = \min\{k \ge 1; kC'(\rho_c) + 2B(\rho_c) < 0\}$.*

Further assume that $A_m(s)$ does not satisfy Assumption A. Then

$$a_i = \frac{\sum_{k=0}^{m\wedge i} \frac{i!}{(i-k)!}(\sim)\oint_{\zeta_c}^{\rho_c} \frac{y^{i-k}D_{m,k}(y)}{A_m(y)} e^{H_m(y)}dy}{(\sim)\oint_{\zeta_c}^{\rho_c} \frac{D_{m,0}(y)}{A_m(y)} e^{H_m(y)}dy}. \tag{5.38}$$

where $H_m(y) = (\sim)\oint_0^y \frac{B_m(s)}{A_m(x)}dx$.

Proof. Again, consider the equation (5.8) with $n = m$, i.e.,

$$A_m(s)G_i^{(m+2)}(s) + B_m(s)G_i^{(m+1)}(s) = U_{mi}(s),$$

but now we view it as a complex-valued differential equation with complex-valued coefficient functions $A_m(s)$ and $B_m(s)$ etc. It is easily seen that this equation does make sense at the above interpretation. Now, solving the above equation, we may still get that

$$G_i^{(m+1)}(s)e^{H_m(s)} - G_i^{(m+1)}(0) = (\sim)\oint_0^s \frac{U_{mi}(y)}{A_m(y)} e^{H_m(y)}dy, \tag{5.39}$$

in the meaning we defined above. By the same reasons as in proving Theorem 5.8, if we let $s \to \rho_c^-$ in (5.39) we immediately obtain

$$-G_i^{(m+1)}(0) = (\sim)\oint_0^{\rho_c} \frac{U_{mi}(y)}{A_m(y)} e^{H_m(y)}dy.$$

Similarly we could get that

$$-G_i^{(m+1)}(0) = (\sim)\oint_0^{\zeta_c} \frac{U_{mi}(y)}{A_m(y)} e^{H_m(y)}dy.$$

By combining the above two equalities, we may, again, obtain

$$(\sim)\oint_{\zeta_c}^{\rho_c} \frac{U_{mi}(y)}{A_m(y)} e^{H_m(y)}dy = 0.$$

Now, substituting (5.28) into the above equality immediately yields (5.38). □

Therefore, in some sense, our Assumption A is just to guarantee that our expression (5.38) could take some simple form. In another word, Assumption A is not a serious limitation but just for the purpose of notational convenience.

Let's consider more details about the case of $m = 2$ under the condition, of course, that $C'(\rho_c) + 2B(\rho_c) > 0$. To this end, we need more information about the properties of the function $B_1(s)$ which are shown below.

Lemma 5.3. *(i) For all $s \in (-1, \rho_b)$, and, in particular, for all $s \in (\zeta_c, \rho_c)$ we have $B(s) > 0$.*

(ii) For all $s \in (\zeta_c, \rho_c)$, we have $C(s) > 0$.

(iii) For all $s \in (-\rho_c, \rho_c)$, and, in particular, for all $s \in [\zeta_c, \rho_c]$ we have $B'(s) < 0$.

(iv) *The function $C'(z)$, viewed as a complex function well-defined at least on the unit disc, has exactly one zero in the disc $\{z; |z| \le \rho_c\}$ and this unique zero, denoted by η_1, is real and non-negative. In particular, we have that for all $s \in [\zeta_c, \eta_1)$, $C'(s) > 0$ while for all $s \in (\eta_1, \rho_c]$, $C'(s) < 0$.*

Proof. Facts (i) and (ii) are easy and, also, have been proved in Lemma 2.1. (iii) is also easy. Indeed, by using the fact that $B'(\rho_c) < 0$ together with $B'(s) = \sum_{j=1}^{\infty} jb_j s^{j-1}$ we obtain $\sum_{j=2}^{\infty} jb_j \rho_c^{j-1} < -b_1$. It follows that for all $z \in \{z; |z| \le \rho_c\}$ we have $\left|\sum_{j=2}^{\infty} jb_j z^{j-1}\right| \le \sum_{j=2}^{\infty} jb_j \rho_c^{j-1} < -b_1$ and thus $B'(z)$ has no zero in the complex disc $\{z; |z| \le \rho_c\}$ and in particular, no zero in the interval $[-\rho_c, \rho_c]$. But $B'(0) = b_1 < 0$ and thus for all $s \in [-\rho_c, \rho_c]$ particularly for $s \in [\zeta_c, \rho_c]$, $B'(s)$ remains the negative sign.

To show (iv), we apply Rouche's Theorem. To this end, define $f(z) = 2c_2 z$ and $g(z) = \sum_{j=1}^{\infty} jc_j z^{j-1} - 2c_2 z$ and thus $f(z) + g(z) = C'(z)$. Now it is easily seen that both $f(z)$ and $g(z)$ are analytic on the open disc $\{z; |z| < \rho_c\}$ and continuous on the circle $\{z; |z| = \rho_c\}$ (In fact, they are even analytic on the unit disc $\{z; |z| < 1\}$). Moreover we could easily prove that on the circle $\{z; |z| = \rho_c\}$ we have

$$|f(z)| > |g(z)| > 0 \tag{5.40}$$

Indeed, firstly, the fact that $|g(z)| = \left|\sum_{j\ne 2} jc_j z^{j-1}\right| > 0$ is clearly true on $\{z; |z| = \rho_c\}$ since all c_j $(j \ne 2)$ are all non-negative and at least one of them is strictly positive. Secondly, since $C'(\rho_c) < 0$ we know that $|g(z)| \le \sum_{j\ne 2} jc_j |z|^{j-1} \le \sum_{j\ne 2} jc_j \rho_c^{j-1} < -2c_2\rho_c = |f(z)|$ and thus (5.40) is proven. Now, by Rouche's Theorem we know that $C'(z)$ and $f(z)$ have the same number of zeros on $\{z; |z| \le \rho_c\}$. But $f(z) = 2c_2 z$ has clearly one zero on $\{z; |z| \le \rho_c\}$ and thus so does $C'(z)$. Finally, considering $C'(0) = c_1 \ge 0$ and $C'(\rho_c) < 0$ and thus this unique zero must be nonnegative. Now, if $c_1 > 0$, then all the conclusions in (iv) have been proven. If $C'(0) = c_1 = 0$, we need to show that for all $s \in [\zeta_c, 0)$ we have $C'(s) > 0$. But this is also easy. Indeed, it follows from the fact that $\lim_{s\uparrow 0^-} \frac{C'(s)}{s} = 2c_2 < 0$.

The following simple lemma now easily follows.

Lemma 5.4. *For $B_1(s) \equiv B(s)\left(B(s) + 1/2C'(s)\right) - 1/2C(s)B'(s)$ we have the following conclusions:*

(i) *For all $s \in [\zeta_c, 0]$ we have $B_1(s) > 0$.*

(ii) *Either $B_1(s) > 0$ for all $s \in [0, \rho_c]$, or $B_1(s)$ has exactly two positive zeros $0 < \lambda_1 < \lambda_2 \le \rho_c$ such that $B_1(s) > 0$ for all $s \in [0, \lambda_1) \cup (\lambda_2, \rho_c]$ and $B_1(s) < 0$ for all $s \in (\lambda_1, \lambda_2)$. Moreover, the latter case happens if and only if* $\min\{B_1(s); s \in [0, \rho_c]\} < 0$.

Proof. Considering $B_1(s) = B(s)\left(B(s) + 1/2C'(s)\right) - 1/2C(s)B'(s)$ and thus (i) immediately follows from Lemma 5.3. For (ii), by Lemma 5.3 we first know that for all $s \in [0, \rho_c]$ we have $-1/2C(s)B'(s) > 0$ together with $B(s) > 0$. Hence for any $s \in (0, \rho_c)$, we have that $B_1(s) < 0$ if and only if

$$B(s) + \frac{1}{2}C'(s) < \frac{C(s)B'(s)}{2B(s)} \tag{5.41}$$

where the right hand side of (5.41) is negative.

But $B(s) + 1/2C'(s)$ is convex function on $[0, 1]$ and thus can have at most two zeros. Since $B(0) + 1/2C'(0) = b_0 + 1/2c_1 > 0$ and thus if $B(s) + 1/2C'(s)$ has no zero or has only one zero on $(0, 1]$, then (5.41) can not be held and thus $B_1(s) > 0$ for all $s \in [0, \rho_c]$.

On the other hand, if $B(s) + 1/2C'(s)$ has two zeros on $(0, 1]$, then in order to make (5.41) true, these two zeros must be all in $(0, \rho_c]$ due to the facts that $B(0) + 1/2C'(0) > 0$ and $C(\rho_c) = 0$. Now the first part of (ii) follows. Finally, if there does exist two zeros $\lambda_1 < \lambda_2$ such that $B_1(s) < 0$ for all $s \in (\lambda_1, \lambda_2)$, then clearly $\min\{B_1(s); s \in [0, \rho_c]\} < 0$ and vice-versa.

Remark 5.5. Lemma 5.4 provides a method to check whether $B_1(s)$ has two zeros or not in $[0, \rho_c]$. But such calculation may be complex since to find the root of $B_1'(s)$ may not be easy. A relatively easy algebra could do as follows: Just first find the zero of $B'(s) + 1/2C''(s)$ on $[0, \rho_c]$ which must be unique, denoted by τ, say. Then calculate $B_1(\tau)$. If $B_1(\tau) < 0$ then we must have $\min\{B_1(s); s \in [0, \rho_c]\} < 0$. But the converse may not be true.

We now have the following conclusion whose proof is essentially given above.

Theorem 5.10. *Suppose that Q is a sub-explosive IBC-q-matrix satisfying both conditions of $C'(\rho_c)+2B(\rho_c) > 0$ and $2C'(\rho_c)+2B(\rho_c) < 0$. Then we have the following conclusions.*

(i) The equation $B(s) + 1/2C'(s) = 0$ either has no root in the interval $(0, \rho_c)$ or has exactly two roots in the interval $(0, \rho_c)$.

(ii) If $B(s) + 1/2C'(s) = 0$ has no root in $(0, \rho_c)$ or if $B(s) + 1/2C'(s) = 0$ does have two roots in $(0, \rho_c)$ but $\min\{B_1(s); s \in [0, \rho_c]\} > 0$, then

$$a_1 = \frac{\int_{\zeta_c}^{\rho_c} \frac{yD_{2,0}(y)+D_{2,1}(y)}{A_2(y)} e^{H_2(y)} dy}{\int_{\zeta_c}^{\rho_c} \frac{D_{2,0}(y)}{A_2(y)} e^{H_2(y)} dy}, \tag{5.42}$$

and for $i \geq 2$,

$$a_i = \frac{\int_{\zeta_c}^{\rho_c} \frac{y^i D_{2,0}(y)+iy^{i-1}D_{2,1}(y)+i(i-1)y^{i-2}D_{2,2}(y)}{A_2(y)} e^{H_2(y)} dy}{\int_{\zeta_c}^{\rho_c} \frac{D_{2,0}(y)}{A_2(y)} e^{H_2(y)} dy}. \tag{5.43}$$

(iii) If $B(s) + 1/2C'(s) = 0$ has two roots in $(0, \rho_c)$ and $\min\{B_1(s); s \in [0, \rho_c]\} < 0$ then

$$a_1 = \frac{(\sim) \oint_{\zeta_c}^{\lambda_2} \frac{yD_{2,0}(y)+D_{2,1}(y)}{A_2(y)} e^{H_2(y)} dy}{(\sim) \oint_{\zeta_c}^{\lambda_2} \frac{D_{2,0}(y)}{A_2(y)} e^{H_2(y)} dy}, \tag{5.44}$$

and for $i \geq 2$

$$a_i = \frac{(\sim) \oint_{\zeta_c}^{\lambda_2} \frac{y^i D_{2,0}(y)+iy^{i-1}D_{2,1}(y)+i(i-1)y^{i-2}D_{2,2}(y)}{A_2(y)} e^{H_2(y)} dy}{(\sim) \oint_{\zeta_c}^{\lambda_2} \frac{D_{2,0}(y)}{A_2(y)} e^{H_2(y)} dy}. \tag{5.45}$$

At the end of this section, we now make the following important remark.

Remark 5.6. The careful readers may have already noticed that nearly all the conclusions obtained in this section are based on the condition $\rho_c < +\infty$ only rather than on the assumption that $0 < C'(1) < +\infty$ as stated in the beginning of this section. Indeed, it is easily seen that Theorems 5.1 to 5.3 and Theorems 5.5 to 5.10 still hold true and all the proofs remain valid if $C'(1) = +\infty$. Even Theorem 5.4 is till true (but proofs need to be revised). In fact the main reason in making the assumption that $0 < C'(1) < +\infty$ is for convenience only. Indeed, if $C'(1) = +\infty$ and $B'(1) = +\infty$, then the uniqueness question has not been fully resolved, see Section 3, and thus we are unable to firmly claim that we are dealing with the irregular case in this section.

6. Examples

Example 6.1. We first give a simple example where the birth structure for both the branching and collision components takes a single birth form. That is we assume that

$$b_0 = a > 0, \quad b_1 = -(a+b), \quad b_2 = b > 0, \quad b_j \equiv 0 \quad (\forall j \geq 3) \tag{6.1}$$

and that

$$c_0 = d > 0, \quad c_1 = r \geq 0, \quad c_2 = -(d+r+c), \quad c_3 = c > 0, \quad c_j \equiv 0 \quad (\forall j \geq 4). \tag{6.2}$$

Note that this special example is just the one considered in Kalinkin (2003). Now using (1.1) we could construct an IBC q-matrix as follows.

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ a & -(a+b) & b & 0 & 0 & \cdots \\ d & 2a+r & -[(d+r+c)+2(a+b)] & c+2b & 0 & \cdots \\ 0 & \binom{3}{2}d & \binom{3}{2}r+3a & -[\binom{3}{2}(d+r+c)+3(a+b)] & \binom{3}{2}c+3b & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Clearly, for this Q, we have

$$B(s) = a-(a+b)s+bs^2 = a(1-s)(1-\frac{bs}{a}) \tag{6.3}$$

and

$$C(s) = d+rs-(d+r+c)s^2+cs^3 = c(s-1)(s-q)(s-\zeta) \tag{6.4}$$

where $q = \frac{(d+r)+\sqrt{(d+r)^2+4dc}}{2c}$ and $\zeta = \frac{(d+r)-\sqrt{(d+r)^2+4dc}}{2c} < 0$. It is easily seen that $C'(1) = c-(2d+r)$ and $B'(1) = b-a$.

By Theorem 4.1 and Corollary 4.1, we know that if $c < (2d+r)$ or if $c = (2d+r)$ and $b \leq a$ then the extinction will definitely occur, that is that for all $i \geq 1$, $a_i = 1$. We now turn our attention to the interesting case of $C'(1) = 0$ together with $B'(1) > 0$ or, equivalently

$c = 2d + r$ and $b > a$. For this case, an easy algebra yields that the discriminant quantity J_0 given in Remark 4.1 takes the form as

$$J_0 = \int_0^1 \frac{A(y)}{C(y)} dy = \frac{1}{d} \int_0^1 [1 + (2 + \frac{r}{d})y]^{\frac{2(bd+ar+2ad)}{(3d+r)(2d+r)}-1} \cdot (1-y)^{\frac{2(b-a)}{3d+r}-2} dy. \tag{6.5}$$

In particular, if $r = 0$, then J_0 takes the very simple form as

$$J_0 = \int_0^1 \frac{A(y)}{C(y)} dy = \frac{1}{d} \int_0^1 (1+2y)^{\frac{2a+b-3d}{3d}} \cdot (1-y)^{-\frac{6d+2a-2b}{3d}} dy. \tag{6.6}$$

Now by (6.5) we can easily get the conclusion that $J_0 < \infty$ if and only if $3d + r < 2(b - a)$ and hence under the condition that $C'(1) = 0$ and $B'(1) > 0$, we know $a_i = 1$ $(\forall i \geq 1)$ if and only if $3d + r \geq 2(b - a)$ which coincides with Corollary 4.3. Note that the case $3d + r = 2(b - a)$ corresponds to the case that $C''(1) = 4B'(1)$ and thus by Corollary 4.3 we still have $a_i = 1$ (for all $i \geq 1$).

Finally we consider the case $C'(1) > 0$ i.e. $c > 2d + r$, and thus $q < 1$. For this case, if $q = \frac{a}{b}$, i.e. if $\frac{a}{b} = \frac{(d+r)+\sqrt{(d+r)^2+4dc}}{2c}$ then $a_i = (\frac{a}{b})^i$ $(i \geq 1)$ while if $a < bq$ then the function $A(y)$ takes the form as

$$A(y) = \left(\frac{q-y}{q}\right)^{\frac{2(bq-a)}{c(q-\zeta)}} \left(\frac{y-\zeta}{-\zeta}\right)^{\frac{2(a-b\zeta)}{c(q-\zeta)}}.$$

Now we have obtained the following conclusion.

Theorem 6.1. *For the IBC-q-matrix determined by (6.1) and (6.2) we have the following conclusions.*

(i) There always exists only one IBCP which is the Feller minimal process and that this Feller minimal process is honest if and only if $c \leq 2d + r$.

(ii) The extinction probabilities $a_i = 1$ $(\forall i \geq 1)$ *if and only if one of the following three cases holds*

(a) $c < (2d + r)$,

(b) $c = (2d + r)$ *and* $b \leq a$,

(c) $c = (2d + r)$, $b > a$ *and* $3d + r \geq 2(b - a)$.

(iii) If $c = (2d + r)$, $b > a$ *and* $3d + r < 2(b - a)$, *then* $a_i < 1$ *and in this case, the extinction probability* a_i *is given by*

$$a_i = \frac{\int_\zeta^1 y^i \left(1 + (2 + \frac{r}{d})y\right)^{\frac{2(bd+ar+2ad)}{(3d+r)(2d+r)}-1} (1-y)^{\frac{2(b-a)}{3d+r}-2} dy}{\int_\zeta^1 \left(1 + (2 + \frac{r}{d})y\right)^{\frac{2(bd+ar+2ad)}{(3d+r)(2d+r)}-1} (1-y)^{\frac{2(b-a)}{3d+r}-2} dy}$$

where $\zeta = -\frac{d}{2d+r}$.

In particular, if $r = 0$, *then* a_i *takes a particular simply form as*

$$a_i = \frac{\int_{-\frac{1}{2}}^1 y^i (1+2y)^{\frac{2a+b}{3d}-1} (1-y)^{\frac{2b-2a-6d}{3d}} dy}{\int_{-\frac{1}{2}}^1 (1+2y)^{\frac{2a+b}{3d}-1} (1-y)^{\frac{2b-2a-6d}{3d}} dy}$$

(iv) If $c > 2d + r$, then $a_i < 1$. Moreover, if $c > 2d + r$ and $\frac{a}{b} = \frac{(d+r)+\sqrt{(d+r)^2+4dc}}{2c}$, then $a_i = \left(\frac{a}{b}\right)^i < 1$ while if $c > 2d + r$ and $\frac{a}{b} < \frac{(d+r)+\sqrt{(d+r)^2+4dc}}{2c}$, then

$$a_i = \frac{\int_\zeta^q \frac{y^i\left(1-\frac{y}{q}\right)^{\alpha-1}\left(1-\frac{y}{\zeta}\right)^{\beta-1}}{1-y}dy}{\int_\zeta^q \frac{\left(1-\frac{y}{q}\right)^{\alpha-1}\left(1-\frac{y}{\zeta}\right)^{\beta-1}}{1-y}dy} \tag{6.7}$$

where q and ζ are given in the line below (6.4) and

$$\alpha = \frac{2(bq-a)}{c(q-\zeta)} \quad \text{and} \quad \beta = \frac{2(a-b\zeta)}{c(q-\zeta)}$$

are two positive constants.

Remark 6.1. If one does the transformation $y = \zeta + (q-\zeta)x$ in (6.7), then the extinction probabilities a_i can be rewritten as

$$a_i = \frac{\sum_{k=0}^{i} \binom{i}{k}(q-\zeta)^k \zeta^{i-k} \int_0^1 \frac{x^k x^{\beta-1}(1-x)^{\alpha-1}}{u-x}dx}{\int_0^1 \frac{x^{\beta-1}(1-x)^{\alpha-1}}{u-x}dx}$$

where $u = \frac{1-\zeta}{q-\zeta} > 1$. In particular,

$$a_1 = \zeta + (q-\zeta)\frac{\int_0^1 \frac{x^{\beta}(1-x)^{\alpha-1}}{u-x}dx}{\int_0^1 \frac{x^{\beta-1}(1-x)^{\alpha-1}}{u-x}dx}.$$

Note also that a_i could be written as linear combinations of beta functions and thus gamma functions. This provides considerable convenience in calculating such extinction probabilities numerically.

Remark 6.2. Conclusion (iv) of the above Theorem 6.1 is the same as that obtained in Kalinkin [10]. In fact, as we are aware, this result is the only result obtained for interacting branching collision processes until now. Note that, however, this result is regarding a special example of the super-explosive case. Even if for this very special example, the subtle sub-explosive case remained unsolved until we tackled this problem. Using our Theorem 5.9 we may easily write down the solution for this sub-explosive case, but we shall not do so here for some obvious reasons.

Note that the extinction properties of IBCPs only depend upon on the two functions $B(s)$ and $C(s)$ and conversely, if we know these two functions, we could easily get the original Q-matrix. For this reason, in the following examples, we shall only specify these two functions rather than the Q-matrix itself.

Example 6.2. Next we provide an example such that $C'(1) = +\infty$ but we keep $B(s)$ as the same in Example 6.1. That is that we consider an IBC q-matrix Q to which the associated $B(s)$ and $C(s)$ are given by

$$B(s) = a - (a+b)s + bs^2 = (1-s)(a-bs)$$

and

$$C(s) = \frac{1}{4} + \frac{s}{2} - \frac{15}{8}s^2 + \sum_{k=3}^{\infty}(a_{k-1} + \frac{a_k}{4})s^k$$

where $a, b > 0$ and $a_k = (1 - \frac{1}{2(k-1)}) \cdot \frac{[2(k-2)]!}{[(k-2)!2^{k-2}]^2}$, $k \geq 2$. An easy algebra shows that

$$C(s) = (s + 1/4)(1 - s)[1 - \frac{s}{\sqrt{1-s}}]$$

and $B'(1) = b - a, C'(1) = +\infty$. Also, $C(s) = 0$ has three roots in $[-1, 1]$, i.e., $-1/4, 1$ and $\rho_c = \frac{\sqrt{5}-1}{2}$ (i.e.ρ_c is the positive root of $s^2 + s - 1 = 0$ in (0,1)). By Theorem 3.1, we know the corresponding IBCP is the dishonest Feller minimal process. Now by the results obtained in the previous sections, we could get that

(i) if $a/b = \rho_c$ then $a_i = {\rho_c}^i$;

(ii) If $a/b < \rho_c$ then

$$a_i = \frac{\int_{-1/4}^{\rho_c} \frac{y^i A(y)}{C(y)} dy}{\int_{-1/4}^{\rho_c} \frac{A(y)}{C(y)} dy}$$

where $A(y) = e^{\int_0^y \frac{2(a-bs)(1-s+s\sqrt{1-s})}{(1/4+s)(1-s-s^2)} ds}$.

Example 6.3. We now provide an example in which we have $B'(1) = +\infty$ together with an arbitrary structure of $C(s)$. Let $b_0 = a > 0$, $b_2 = \frac{b}{2} > 0$, $b_{2n} = b \cdot \frac{(2n-3)!!}{(2n)!!}$ $(n \geq 2)$ together with $b_{2n+1} = 0$ $(\forall n \geq 1)$ and $b_1 = -(a + b)$. Then

$$B(s) = a - (a + b)s + \frac{b}{2}s^2 + b\sum_{n=2}^{\infty} \frac{(2n-3)!!}{(2n)!!} s^{2n}$$

An easy algebra shows that $B(s)$ has the following form

$$B(s) = (1 - s) \left[a + b - b\sqrt{\frac{1+s}{1-s}} \right] \tag{6.8}$$

which has two positive zeros 1 and ρ_b as

$$\rho_b = \frac{(a + b)^2 - b^2}{(a + b)^2 + b^2} \tag{6.9}$$

together with the obvious fact that $B'(1) = +\infty$. Further assume that $C(s) = \sum_{n=0}^{\infty} c_n s^n$ is in a general form with zeros 1 and possibly ρ_c. By using the results obtained in the previous sections, we could obtain the following conclusion.

Theorem 6.2. *Let Q be an IBC q-matrix to which the associated $B(s)$ is given in (6.8) together with the general form of $C(s) = \sum_{n=0}^{\infty} c_n s^n$. Then we have* *(i) if $C'(1) < 0$, then $a_i = 1$,*

(ii) while if $0 < C'(1) < +\infty$, then $a_i < 1$. Moreover, if $\frac{(a+b)^2-b^2}{(a+b)^2+b^2} < \rho_c$ then

$$a_i = \frac{\int_{\zeta_c}^{\rho_c} \frac{y^i g(y)A(y)}{2(1-y)}dx}{\int_{\zeta_c}^{\rho_c} \frac{g(y)A(y)}{2(1-y)}dy} \tag{6.10}$$

where

$$A(y) = \exp\left\{(a+b)\int_0^y g(x)dx - b\int_0^y g(x)\sqrt{\frac{1+x}{1-x}}dx\right\}, \tag{6.11}$$

$g(y) = \frac{C(y)}{2(1-y)}$ is a bounded and analytic function of y on $[0,1]$ and $\zeta_c \in (-1,0)$ and $\rho_c \in (0,1)$ are zeros of $C(s)$.

Proof. By (6.8), it is easily seen that $B(s) \sim -2b(1-s)^{\frac{1}{2}}$ (as $s \to 1^-$) and thus (i) immediately follows from Corollary 4.2 and the fact that $C'(1) < 0$. Fact (ii) follows from Theorem 5.3 directly. □

Finally, we provide an example in which the structures of $B(s)$ and $C(s)$ are very similar. We are interested to see what the effects are when $B(s)$ and $C(s)$ "compete" in some sense.

Example 6.4. Let Q be an IBC q-matrix to which the associated $B(s)$ and $C(s)$ are given by

$$B(s) = a - (a+b)s + b(1-p)s^2\sum_{k=0}^{\infty}(ps)^k \tag{6.12}$$

and

$$C(s) = d - (d+c)s^2 + c(1-q)s^3\sum_{k=0}^{\infty}(qs)^k \tag{6.13}$$

where a, b, c and d are positive numbers and $0 < p < 1$ and $0 < q < 1$. An easy algebra shows that

$$B(s) = \frac{(1-s)[a-(ap+b)s]}{1-ps} = (1-s)[a - \frac{bs}{1-ps}] \tag{6.14}$$

and that

$$C(s) = (1-s)\left[d + ds - \frac{cs^2}{1-qs}\right] \tag{6.15}$$

or

$$C(s) = \frac{(dq+c)(1-s)(\rho_c - s)(s-\zeta)}{1-qs} \tag{6.16}$$

with

$$\rho_c = \frac{d(1-q) + \sqrt{d^2(1-q)^2 + 4d(dq+c)}}{2(dq+c)} \tag{6.17}$$

and

$$\zeta = \frac{d(1-q) - \sqrt{d^2(1-q)^2 + 4d(dq+c)}}{2(dq+c)}. \tag{6.18}$$

By (6.14), we know that

$$\rho_b = \frac{a}{ap+b}. \tag{6.19}$$

and that

$$B'(1) = \frac{b}{1-p} - a \tag{6.20}$$

and thus $B'(1) > 0$ if and only if $1-p < \frac{b}{a}$.

Some other key quantities could be easily calculated as

$$C'(1) = \frac{c}{1-q} - 2d \tag{6.21}$$

and

$$C''(1) = 4c - 2d + \frac{2cq(3-2q)}{(1-q)^2} < \infty. \tag{6.22}$$

Under the condition $C'(1) = 0$, then $C''(1)$ could be further simplified as

$$C''(1) = \frac{c(3-q)}{(1-q)^2}. \tag{6.23}$$

Now applying the results obtained before to this example we could easily get the following conclusion.

Theorem 6.3. *For the IBCP determined by the IBC-q-matrix Q associated with (6.12) and (6.13), we have*

(i) the Q is regular, i.e. the IBCP is honest if and only if $1-q \geq \frac{c}{2d}$.

(ii) For all $i \geq 1$, the extinction probabilities $a_i = 1$ if and only if one of the following conditions hold

(a) $1-q > \frac{c}{2d}$;

(b) $1-q = \frac{c}{2d}$ and $1-p \geq \frac{b}{a}$;

(c) $1-q = \frac{c}{2d}$, $1-p < \frac{b}{a}$ and $\frac{3-q}{4(1-q)^2} + \frac{a}{c} \geq \frac{b}{c(1-p)}$.

(iii) If $1-q = \frac{c}{2d}$, $1-p < \frac{b}{a}$ and $\frac{3-q}{4(1-q)^2} + \frac{a}{c} < \frac{b}{c(1-p)}$, then

$$a_i = \frac{\int_\zeta^1 \frac{y^i(1-qy)A(y)}{(1-y)^2(y-\zeta)}dy}{\int_\zeta^1 \frac{(1-qy)A(y)}{(1-y)^2(y-\zeta)}dy}$$

where

$$A(y) = \exp\left\{\frac{2}{dq+c}\int_0^y \frac{(1-qx)[a-(ap+b)x]}{(1-px)(1-x)(x-\zeta)}dy\right\}$$

and ζ is given in (6.18).

(iv) Suppose $1-q < \frac{c}{2d}$. Then if $\rho_c = \rho_b$ then

$$a_i = \left(\frac{a}{ap+b}\right)^i \quad (i \geq 1)$$

while if $\rho_b < \rho_c$*, then*

$$a_i = \frac{\int_\zeta^{\rho_c} \frac{y^i(1-qy)A(y)}{(1-y)(\rho_c-y)(y-\zeta)}dy}{\int_\zeta^{\rho_c} \frac{(1-qy)A(y)}{(1-y)(\rho_c-y)(y-\zeta)}dy}$$

where

$$A(y) = \exp\left\{\frac{2}{dq+c} \cdot \int_0^y \frac{(1-qx)[a-(ap+b)x]}{(1-px)(\rho_c-x)(x-\zeta)}dx\right\}$$

and ρ_c *and* ζ *are given in (6.17) and (6.18), respectively.*

For this example, we are particularly interested in the so-called "perfectly competitively" situation in the sense that $p = q$, $a = d$ and $b = c$. For this special case, we first note that $C'(1) = 0$ implies both $B'(1) = a > 0$ and $C''(1) > 4B'(1)$ and thus the extinction probabilities $a_i = 1$ for all $i \geq 1$ so long as $C'(1) = 0$. Secondly, $C'(1) > 0$ implies $B'(1) > 0$ and thus $\rho_b < 1$. Indeed, $C'(1) > 0$ is equivalent to $\frac{b}{1-p} > 2a$ but $B'(1)$ is just $\frac{b}{1-p} - a$. Moreover, we could further show that $C'(1) > 0$ implies $0 < \rho_b < \rho_c < 1$. In fact, an easy algebra could show that $\rho_b < \rho_c$ is equivalently to the trivial fact that $4ab > 0$. Now we could get the following satisfactory conclusion which shows that in the evolution of this IBCP the collision component totally dominates the branching component.

Corollary 6.1. *For the IBCP process given in Example 6.4 with the further conditions that* $0 < p = q < 1, a = d > 0$*, and* $b = c > 0$*. Then the following statements are equivalent:*

(i) The IBC-q-matrix is regular, i.e. the IBCP is honest.

(ii) The extinction probabilities are all 1, i.e. $a_i = 1$ *for all* $i \geq 1$*.*

(iii) $1 - p \geq \frac{b}{2a}$

Furthermore, if $1 - p < \frac{b}{2a}$, then the IBCP is dishonest and the extinction probability starting from $i \geq 1$ is less than one and these extinction probabilities are given by

$$a_i = \frac{\int_\zeta^{\rho_c} \frac{y^i(1-py)A(y)}{(1-y)[a(1+y)(1-py)-by^2]}dy}{\int_\zeta^{\rho_c} \frac{(1-py)A(y)}{(1-y)[a(1+y)(1-py)-by^2]}dy}$$

where

$$A(y) = \exp\left\{2\int_0^y \frac{a(1-px)-bx}{a(1+x)(1-px)-bx^2}dx\right\}$$

and ρ_c and ζ are the positive and negative roots of the equation $a(1+y)(1-py) - by^2 = 0$, respectively.

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