The Ruin Probability of the Renewal Model with Constant Interest Force and Negatively Dependent Heavy-tailed Claims

Yiqing Chen *, Kai W. Ng
The Department of Statistics and Actuarial Science
The University of Hong Kong
Pokfulam Road, Hong Kong

May 21, 2008

Abstract

Recently, Tang (2005, Scand. Actuar. J., no. 1, 1–5) obtained a simple asymptotic formula for the ruin probability of the renewal risk model with constant interest force and regularly varying tailed claims. In this paper, we use a completely different approach to extend Tang's result to the case in which the claims are pairwise negatively dependent and extended regularly varying tailed.

Keywords: Asymptotics, constant interest force, negative dependence, extended regular variation, renewal model, ruin probability.

1 The model

We investigate the ruin probability of a nonstandard renewal model. In this model the claims, X_n, $n = 1, 2, \ldots$, form a sequence of identically distributed, not necessarily independent, and nonnegative random variables with common distribution function F, and their inter-arrival times, Y_n, $n = 1, 2, \ldots$, form another sequence of independent, identically distributed (i.i.d.), and nonnegative random variables, which are independent of the random variables X_n, $n = 1, 2, \ldots$, and, of course, are not degenerate at 0. The locations of the successive claims, $\tau_n = \sum_{k=1}^{n} Y_k$, constitute a renewal counting process

$$N_t = \operatorname{card}\{n = 1, 2, \ldots : \tau_n \le t\}, \qquad t \ge 0,$$

*Corresponding author: Yiqing Chen; tel.: +852-2859 1920; fax: +852-2858 9041. E-mail addresses: yqchen@hkusua.hku.hk (Yiqing Chen), kaing@hku.hk (Kai W. Ng).

where, by convention, the cardinality of the empty set is 0. Denote by m_t the mean function (renewal measure) of N_t, $t \geq 0$. It is well known that

$$m_t = \sum_{n=1}^{\infty} \Pr(\tau_n \leq t), \qquad t \geq 0. \tag{1.1}$$

The total amount of claims accumulated by time $t \geq 0$ is represented as a compound sum

$$S_t = \sum_{n=1}^{N_t} X_n, \qquad t \geq 0,$$

where the summation over an empty set of index values is considered to be 0. Let c be the constant premium rate, let $\delta > 0$ be the constant interest force (that is, after time t a capital x becomes $x\mathrm{e}^{\delta t}$), and let $x \geq 0$ be the initial surplus of the insurance company. Then the total surplus up to time $t \geq 0$, denoted by U_t, satisfies the equation

$$U_t = x\mathrm{e}^{\delta t} + c\int_0^t \mathrm{e}^{\delta(t-y)}\mathrm{d}y - \sum_{n=1}^{\infty} X_n \mathrm{e}^{\delta(t-\tau_n)} 1_{(\tau_n \leq t)}, \tag{1.2}$$

where 1_A denotes the indicator function of an event A; see Tang (2005b). The ruin probability is defined by

$$\psi(x) = \Pr\left(U_t < 0 \text{ for some } t \geq 0 \mid U_0 = x\right). \tag{1.3}$$

2 Introduction and the main result

Hereafter, all limit relationships are for $x \to \infty$ unless otherwise stated; for two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$, write $a(x) \lesssim b(x)$ if $\limsup a(x)/b(x) \leq 1$, and write $a(x) \gtrsim b(x)$ if $\liminf a(x)/b(x) \geq 1$.

We say that a distribution F belongs to the class $\mathcal{R}$ if there is some $\alpha > 0$ such that

$$\lim_{x\to\infty} \frac{\overline{F}(sx)}{\overline{F}(x)} = s^{-\alpha} \qquad \text{for all } s > 0. \tag{2.1}$$

In this case we write $F \in \mathcal{R}_{-\alpha}$.

A bit larger class is the so-called Extended Regular Variation (ERV) class. By definition, a distribution F belongs to the class ERV if there are some $0 < \alpha \leq \beta < \infty$ such that

$$s^{-\beta} \leq \liminf_{x\to\infty} \frac{\overline{F}(sx)}{\overline{F}(x)} \leq \limsup_{x\to\infty} \frac{\overline{F}(sx)}{\overline{F}(x)} \leq s^{-\alpha} \qquad \text{for all } s \geq 1, \tag{2.2}$$

or, equivalently,

$$s^{-\alpha} \leq \liminf_{x\to\infty} \frac{\overline{F}(sx)}{\overline{F}(x)} \leq \limsup_{x\to\infty} \frac{\overline{F}(sx)}{\overline{F}(x)} \leq s^{-\beta} \qquad \text{for all } 0 < s \leq 1. \tag{2.3}$$

In this case we write $F \in \mathrm{ERV}(-\alpha, -\beta)$.

Note that the class $\mathcal{R}$ is the union of all $\mathcal{R}_{-\alpha}$ over the range $0 < \alpha < \infty$ and the class ERV is the union of all $\mathrm{ERV}(-\alpha, -\beta)$ over the range $0 < \alpha \le \beta < \infty$. For more detail of these two distribution classes, the reader is referred to the monograph Bingham et al. (1987).

The asymptotic behavior of the ruin probability $\psi(x)$ with F heavy tailed has been extensively investigated in the literature. Under the condition $F \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$, starting from an integral equation of Sundt and Teugels (1995), Klüppelberg and Stadtmüller (1998) obtained the relation

$$\psi(x) \sim \frac{\lambda}{\alpha\delta} \overline{F}(x). \tag{2.4}$$

Using the reflected random walk theory, Asmussen (1998) extended the study to a larger class of heavy-tailed distributions and obtained the relation

$$\psi(x) \sim \frac{\lambda}{\delta} \int_x^\infty \frac{\overline{F}(y)}{y} \mathrm{d}y; \tag{2.5}$$

see Corollary 4.1(ii) of his paper. Complementary discussions can be found in Kalashnikov and Konstantinides (2000) and Konstantinides et al. (2002), who also started from the same integral equation of Sundt and Teugels (1995) and reexamined relation (2.5). Applying Karamata's theorem, we easily see that when $F \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$, relation (2.5) coincides with relation (2.4). Recently, using a result of Resnick and Willekens (1991), Tang (2005a) found a simple method to extend relation (2.4) to the renewal model. See also Tang (2005b) for a result similar to (2.5) for the finite-time ruin probability.

We would like to remark that the methods used in all the references mentioned above heavily rely on the i.i.d. assumption on the claims, while the methods of Klüppelberg and Stadtmüller (1998) and Tang (2005a) also rely on the regular variation assumption (2.1) of the claimsize distribution.

In this paper we use a different, purely probabilistic, method to further extend the study to the case that the claims are pairwise negatively dependent with common distribution belonging to the class ERV.

We say that two random variables, X_1 and X_2, are Negatively Dependent (ND) if for all real numbers x_1, x_2,

$$\Pr(X_1 \le x_1, X_2 \le x_2) \le \Pr(X_1 \le x_1) \Pr(X_2 \le x_2),$$

or, equivalently,

$$\Pr(X_1 > x_1, X_2 > x_2) \le \Pr(X_1 > x_1) \Pr(X_2 > x_2).$$

See, for instance, Lehmann (1966). We say that a sequence of random variables $\{X_1, X_2, \ldots\}$ is pairwise ND if for all positive integers $i \ne j$ the random variables X_i and X_j are ND.

The so-called Farlie-Gumbel-Morgenstern family of distributions provides a simple mechanism to construct practically interesting pairs of random variables that are ND but not independent. This family of distributions has the form

$$F_{X_1,X_2}(x_1,x_2) = F_1(x_1)F_2(x_2)\left(1 - a\overline{F_1}(x_1)\overline{F_2}(x_2)\right),$$

where F_1 and F_2 are corresponding marginal distributions and a is a positive constant. We refer the reader to Kotz et al. (2000) for more details.

Our main result is the following:

Theorem 1. *Consider the renewal model introduced in Section 1. If the claims X_n, $n = 1, 2, \ldots$, are pairwise ND with common distribution $F \in \mathrm{ERV}$, then*

$$\psi(x) \sim \int_0^\infty \overline{F}(x\mathrm{e}^{\delta t})\mathrm{d}m_t. \tag{2.6}$$

In particular, if $N(\cdot)$ is a homogenous Poisson process with intensity $\lambda > 0$, relation (2.6) reduces to

$$\psi(x) \sim \lambda \int_0^\infty \overline{F}(x\mathrm{e}^{\delta t})\mathrm{d}t = \frac{\lambda}{\delta}\int_x^\infty \frac{\overline{F}(y)}{y}\mathrm{d}y, \tag{2.7}$$

which coincides with relation (2.5). Explicit forms of the renewal function m_t are also available when the inter-arrival times have a uniform distribution or a matrix-exponential distribution (of which exponential and Erlang distributions are special cases); see Asmussen and Bladt (1997).

Furthermore, from relation (2.7) we see that the asymptotic behavior of the ruin probability is insensitive to the pairwise ND structure among the heavy-tailed claims. A similar phenomena was observed by Kaas and Tang (2005) when they investigated precise large deviation probabilities of sums of heavy-tailed and Negatively Cumulative Dependent (NCD) random variables. As interpreted by Kaas and Tang (2005), a sequence of random variables $\{Z_1, Z_2, \ldots,\}$ is said to be NCD if for every finite subset $\mathcal{I}$ of $\{1, 2, \ldots,\}$ and every $k \in \{1, 2, \ldots,\} - \mathcal{I}$, the sum $S_{\mathcal{I}} = \sum_{i\in\mathcal{I}} Z_i$ and the random variable Z_k are ND. Clearly, NCD is a special case of pairwise ND.

3 Preliminaries

Let $F \in \mathrm{ERV}(-\alpha, -\beta)$ for some $0 < \alpha \le \beta < \infty$. From Proposition 2.2.1 of Bingham et al. (1987) or Section 3.3 of Tang and Tsitsiashvili (2003), we know that, for any α' and β', $\alpha' < \alpha$, $\beta' > \beta$, there are positive constants C_i and D_i, $i = 1, 2$, such that the inequality

$$\frac{\overline{F}(y)}{\overline{F}(x)} \ge C_1 \left(x/y\right)^{\alpha'} \tag{3.1}$$

holds whenever $x \geq y \geq D_1$, and that the inequality

$$\frac{\overline{F}(y)}{\overline{F}(x)} \leq C_2 \left(x/y\right)^{\beta'} \tag{3.2}$$

holds whenever $x \geq y \geq D_2$. Furthermore, due to the arbitrariness of $\beta' > \beta$, fixing the variable y in (3.2) leads to

$$x^{-\beta^*} = o\left(\overline{F}(x)\right) \qquad \text{for } \beta^* > \beta. \tag{3.3}$$

Recall relation (2.2) or (2.3). It is trivial that if $F \in \mathrm{ERV}$ then the relation

$$\overline{F}(x+l) \sim \overline{F}(x) \tag{3.4}$$

holds for all real numbers l. Relation (3.4) defines the class $\mathcal{L}$ of long-tailed distributions.

The following first lemma shows a sufficient condition for the product of independent random variables to have a distribution belonging to the class ERV. For more details we refer the reader to Cline and Samorodnitsky (1994), who extensively investigated the subexponentiality of product of two independent random variables.

Lemma 1. *Let X and Y be two independent and nonnegative random variables with distributions F and G, respectively. If $F \in \mathrm{ERV}(-\alpha, -\beta)$ for some $0 < \alpha \leq \beta < \infty$ and Y is bounded but not degenerate at 0, then the distribution of the product XY still belongs to the class $\mathrm{ERV}(-\alpha, -\beta)$.*

Proof. Let Y be bounded from above by a constant $C > 0$. For each $s \geq 1$, recalling relation (2.2) we have

$$\begin{aligned}
\limsup_{x\to\infty} \frac{\Pr\left(XY > sx\right)}{\Pr\left(XY > x\right)} &= \limsup_{x\to\infty} \frac{\int_0^C \overline{F}(sx/y)\mathrm{d}G(t)}{\int_0^C \overline{F}(x/y)\mathrm{d}G(t)} \\
&\leq \limsup_{x\to\infty} \sup_{0<y\leq C} \frac{\overline{F}(sx/y)}{\overline{F}(x/y)} = \limsup_{x\to\infty} \frac{\overline{F}(sx)}{\overline{F}(x)} \leq s^{-\alpha}.
\end{aligned}$$

Similarly, the relation

$$\liminf_{x\to\infty} \frac{\Pr\left(XY > sx\right)}{\Pr\left(XY > x\right)} \geq s^{-\beta}$$

also holds. Thus, the distribution of XY belongs to the class $\mathrm{ERV}(-\alpha, -\beta)$. □

The following second lemma will play a crucial role in the proof of Theorem 1:

Lemma 2. *Consider the renewal model introduced in Section 1. Under the conditions of Theorem 1, for any positive integer m_0 we have*

$$\Pr\left(\sum_{n=1}^{m_0} X_n \mathrm{e}^{-\delta\tau_n} > x\right) \sim \sum_{n=1}^{m_0} \Pr\left(X_n \mathrm{e}^{-\delta\tau_n} > x\right). \tag{3.5}$$

Proof. Relation (3.5) is trivial if $m_0 = 1$. Herewith we assume $m_0 \geq 2$. Let $1 \leq i \neq j \leq m_0$ and $x > 0$. By the pairwise ND property of $\{X_n, n = 1, 2, \ldots\}$ and the independence between $\{X_n, n = 1, 2, \ldots\}$ and $\{\tau_n, n = 1, 2, \ldots\}$, we have

$$\begin{aligned}\Pr\left(X_i \mathrm{e}^{-\delta\tau_i} > x, X_j \mathrm{e}^{-\delta\tau_j} > x\right) &\leq \Pr\left(X_i \mathrm{e}^{-\delta\tau_1} > x, X_j > x\right) \\ &\leq \Pr\left(X_i \mathrm{e}^{-\delta\tau_1} > x\right) \overline{F}(x) = o\left(\Pr\left(X_1 \mathrm{e}^{-\delta\tau_1} > x\right)\right).\end{aligned}$$

Now we are ready to prove (3.5). On the one hand, we have

$$\begin{aligned}\Pr\left(\sum_{n=1}^{m_0} X_n \mathrm{e}^{-\delta\tau_n} > x\right) &\geq \Pr\left(\bigcup_{n=1}^{m_0} \left(X_n \mathrm{e}^{-\delta\tau_n} > x\right)\right) \\ &\geq \sum_{n=1}^{m_0} \Pr\left(X_n \mathrm{e}^{-\delta\tau_n} > x\right) - \sum_{1\leq i\neq j\leq m_0} \Pr\left(X_i \mathrm{e}^{-\delta\tau_i} > x, X_j \mathrm{e}^{-\delta\tau_j} > x\right) \\ &\sim \sum_{n=1}^{m_0} \Pr\left(X_n \mathrm{e}^{-\delta\tau_n} > x\right). \end{aligned} \tag{3.6}$$

On the other hand, for an arbitrarily fixed number $L > 0$ we find that

$$\begin{aligned}&\Pr\left(\sum_{n=1}^{m_0} X_n \mathrm{e}^{-\delta\tau_n} > x\right) \\ &\leq \Pr\left(\bigcup_{n=1}^{m_0} \left(X_n \mathrm{e}^{-\delta\tau_n} > x - L\right)\right) + \Pr\left(\sum_{n=1}^{m_0} X_n \mathrm{e}^{-\delta\tau_n} > x, \bigcap_{i=1}^{m_0} \left(X_i \mathrm{e}^{-\delta\tau_i} \leq x - L\right)\right) \\ &= I_1(x, L) + I_2(x, L).\end{aligned} \tag{3.7}$$

First we deal with $I_1(x, L)$. Since by Lemma 1, for every $n = 1, 2, \ldots$ the distribution of the product $X_n \mathrm{e}^{-\delta\tau_n}$ belongs to the class ERV (hence is long tailed), we have

$$I_1(x, L) \leq \sum_{n=1}^{m_0} \Pr\left(X_n \mathrm{e}^{-\delta\tau_n} > x - L\right) \sim \sum_{n=1}^{m_0} \Pr\left(X_n \mathrm{e}^{-\delta\tau_n} > x\right). \tag{3.8}$$

For $I_2(x, L)$, by the pairwise ND property once again,

$$\begin{aligned}I_2(x, L) &= \Pr\left(\sum_{n=1}^{m_0} X_n \mathrm{e}^{-\delta\tau_n} > x, \frac{x}{m_0} < \max_{1\leq i\leq m_0} X_i \mathrm{e}^{-\delta\tau_i} \leq x - L\right) \\ &\leq \sum_{i=1}^{m_0} \Pr\left(\sum_{n=1}^{m_0} X_n \mathrm{e}^{-\delta\tau_n} - X_i \mathrm{e}^{-\delta\tau_i} > L, X_i \mathrm{e}^{-\delta\tau_i} > \frac{x}{m_0}\right) \\ &\leq \sum_{i=1}^{m_0} \sum_{n:1\leq n\leq m_0, n\neq i} \Pr\left(X_n \mathrm{e}^{-\delta\tau_n} > \frac{L}{m_0 - 1}, X_i \mathrm{e}^{-\delta\tau_i} > \frac{x}{m_0}\right) \\ &\leq \sum_{i=1}^{m_0} \sum_{n:1\leq n\leq m_0, n\neq i} \Pr\left(X_n > \frac{L}{m_0 - 1}, X_i \mathrm{e}^{-\delta\tau_1} > \frac{x}{m_0}\right) \\ &\leq m_0(m_0 - 1) \Pr\left(X_1 > \frac{L}{m_0 - 1}\right) \Pr\left(X_1 \mathrm{e}^{-\delta\tau_1} > \frac{x}{m_0}\right).\end{aligned}$$

By the last inequality of (2.3), it is easy to see that for some $C(m_0) > 0$, the inequality

$$\Pr\left(X_1 \mathrm{e}^{-\delta\tau_1} > x/m_0\right) \le C(m_0) \Pr\left(X_1 \mathrm{e}^{-\delta\tau_1} > x\right)$$

holds for all $x > 0$. Therefore,

$$\lim_{L\to\infty} \limsup_{x\to\infty} \frac{I_2(x,L)}{\sum_{n=1}^{m_0} \Pr\left(X_n \mathrm{e}^{-\delta\tau_n} > x\right)} = 0. \tag{3.9}$$

From (3.7), (3.8), and (3.9), we conclude that

$$\Pr\left(\sum_{n=1}^{m_0} X_n \mathrm{e}^{-\delta\tau_n} > x\right) \lesssim \sum_{n=1}^{m_0} \Pr\left(X_n \mathrm{e}^{-\delta\tau_n} > x\right). \tag{3.10}$$

By (3.6) and (3.10) we obtain (3.5). □

4 Proof of Theorem 1

Following Tang (2005a,b), we define the discounted values of the surplus process (1.2) as

$$\widetilde{U}_t = \mathrm{e}^{-\delta t} U_t = x + c\int_0^t \mathrm{e}^{-\delta y} \mathrm{d}y - \sum_{n=1}^{\infty} X_n \mathrm{e}^{-\delta\tau_n} 1_{(\tau_n \le t)}, \qquad t \ge 0,$$

from which we see that

$$x - \sum_{n=1}^{\infty} X_n \mathrm{e}^{-\delta\tau_n} \le \widetilde{U}_t \le x + c/\delta - \sum_{n=1}^{\infty} X_n \mathrm{e}^{-\delta\tau_n} 1_{(\tau_n \le t)}, \qquad t \ge 0.$$

From (1.3) we have

$$\psi(x) = \Pr\left(\widetilde{U}_t < 0 \text{ for some } t \ge 0 \mid \widetilde{U}_0 = x\right).$$

It follows that

$$\Pr\left(\sum_{n=1}^{\infty} X_n \mathrm{e}^{-\delta\tau_n} > x + c/\delta\right) \le \psi(x) \le \Pr\left(\sum_{n=1}^{\infty} X_n \mathrm{e}^{-\delta\tau_n} > x\right). \tag{4.1}$$

If we have proven the relation

$$\Pr\left(\sum_{n=1}^{\infty} X_n \mathrm{e}^{-\delta\tau_n} > x\right) \sim \int_0^\infty \overline{F}(x\mathrm{e}^{\delta t}) \mathrm{d}m_t, \tag{4.2}$$

then for any $\epsilon > 0$, we find that

$$\begin{aligned}
\Pr\left(\sum_{n=1}^{\infty} X_n \mathrm{e}^{-\delta\tau_n} > x + c/\delta\right) &\sim \int_0^\infty \overline{F}\left((x + c/\delta)\,\mathrm{e}^{\delta t}\right) \mathrm{d}m_t \\
&\gtrsim \int_0^\infty \overline{F}\left((1+\epsilon)x\mathrm{e}^{\delta t}\right) \mathrm{d}m_t \\
&\gtrsim (1+\epsilon)^{-\beta} \int_0^\infty \overline{F}\left(x\mathrm{e}^{\delta t}\right) \mathrm{d}m_t,
\end{aligned}$$

where the last step is due to the first inequality of (2.2). By the arbitrariness of $\epsilon > 0$, it follows that

$$\Pr\left(\sum_{n=1}^{\infty} X_n \mathrm{e}^{-\delta\tau_n} > x + c/\delta\right) \gtrsim \int_0^{\infty} \overline{F}\left(x\mathrm{e}^{\delta t}\right) \mathrm{d}m_t. \tag{4.3}$$

Hence, we conclude from (4.1), (4.2), and (4.3) that relation (2.6) holds. In this way, we will have completed the proof of Theorem 1.

It remains to prove relation (4.2). Write

$$\Delta_m = \sum_{n=m+1}^{\infty} X_n \mathrm{e}^{-\delta\tau_n} \qquad \text{for } m = 0, 1, \ldots.$$

We follow the proof of Theorem 3.1 of Tang and Tsitsiashvili (2004), but in a simpler manner, to show that $\Pr\left(\Delta_m > x\right)$ is asymptotically negligible in comparison to $\Pr\left(X_1 \mathrm{e}^{-\delta\tau_1} > x\right)$ as x and m become sufficiently large. For all integers m such that $\sum_{n=m+1}^{\infty} n^{-2} < 1$, we derive that

$$\begin{aligned}
\Pr\left(\Delta_m > x\right) &\le \Pr\left(\sum_{n=m+1}^{\infty} X_n \mathrm{e}^{-\delta\tau_n} > \sum_{n=m+1}^{\infty} \frac{x}{n^2}\right) \\
&\le \Pr\left(\bigcup_{n=m+1}^{\infty} \left(X_n \mathrm{e}^{-\delta\tau_n} > \frac{x}{n^2}\right)\right) \\
&\le \sum_{n=m+1}^{\infty} \Pr\left(X_n \mathrm{e}^{-\delta\tau_n} > \frac{x}{n^2}\right).
\end{aligned} \tag{4.4}$$

Choose some α' and β', $0 < \alpha' < \alpha \le \beta < \beta' < \infty$. Then, there are positive constants C_i and D_i, $i = 1, 2$, such that inequalities (3.1) and (3.2) hold accordingly. For all $n = 1, 2, \ldots$ and $x > 0$, introduce the events $A_1(n, x) = \left(n^{-2}\mathrm{e}^{\delta\tau_n} \le D_2/x\right)$, $A_2(n, x) = \left(D_2/x < n^{-2}\mathrm{e}^{\delta\tau_n} \le 1\right)$, and $A_3(n, x) = \left(n^{-2}\mathrm{e}^{\delta\tau_n} > 1\right)$. We divide the right-hand side of (4.4) into three parts as $I_1(m, x) + I_2(m, x) + I_3(m, x)$ with

$$I_k(m, x) = \sum_{n=m+1}^{\infty} \mathrm{E}\left[\Pr\left(X_n > n^{-2}\mathrm{e}^{\delta\tau_n} x \,\middle|\, \tau_n\right) 1_{A_k(n,x)}\right] \qquad \text{for } k = 1, 2, 3.$$

Applying Chebyshev's inequality and relation (3.3), we have

$$I_1(m, x) \le \sum_{n=m+1}^{\infty} \Pr\left(A_1(n, x)\right) \le \left(\frac{x}{D_2}\right)^{-\beta'} \sum_{n=1}^{\infty} n^{2\beta'} \left(\mathrm{E}\left[\mathrm{e}^{-\delta\beta'\tau_1}\right]\right)^n = o\left(\overline{F}(x)\right).$$

Applying inequalities (3.2) and (3.1), for all $x \ge \max\{D_1, D_2\}$, we obtain, respectively,

$$I_2(m, x) \le C_2 \overline{F}(x) \sum_{n=m+1}^{\infty} \mathrm{E}\left[n^{2\beta'} \mathrm{e}^{-\delta\beta'\tau_n} 1_{A_2(n,x)}\right] \le C_2 \overline{F}(x) \sum_{n=m+1}^{\infty} n^{2\beta'} \left(\mathrm{E}\left[\mathrm{e}^{-\delta\beta'\tau_1}\right]\right)^n,$$

and

$$I_3(m,x) \leq \frac{\overline{F}(x)}{C_1} \sum_{n=m+1}^{\infty} \mathrm{E}\left[\left(n^{-2}\mathrm{e}^{\delta\tau_n}\right)^{-\alpha'} 1_{A_3(n,x)}\right] \leq \frac{\overline{F}(x)}{C_1} \sum_{n=m+1}^{\infty} n^{2\alpha'} \left(\mathrm{E}\left[\mathrm{e}^{-\delta\alpha'\tau_1}\right]\right)^n.$$

Hence,

$$\lim_{m\to\infty} \limsup_{x\to\infty} \frac{I_k(m,x)}{\overline{F}(x)} = 0 \qquad \text{for } k=2,3.$$

Substituting these results into (4.4) leads to

$$\lim_{m\to\infty} \limsup_{x\to\infty} \frac{\Pr\left(\Delta_m > x\right)}{\overline{F}(x)} = \lim_{m\to\infty} \limsup_{x\to\infty} \sum_{n=m+1}^{\infty} \frac{\Pr\left(X_n \mathrm{e}^{-\delta\tau_n} > xn^{-2}\right)}{\overline{F}(x)} = 0. \tag{4.5}$$

For arbitrarily fixed $y_0 > 0$ such that $p_0 = \Pr(\tau_1 \leq y_0) > 0$, by relation (2.2), it is easy to see that

$$\Pr\left(X_1 \mathrm{e}^{-\delta\tau_1} > x\right) \geq p_0 \Pr\left(X_1 \mathrm{e}^{-\delta y_0} > x\right) \gtrsim p_0 \mathrm{e}^{-\delta y_0 \beta} \overline{F}(x).$$

Therefore by (4.5), for arbitrarily fixed $0 < \varepsilon < 1$, there are some integer m_0 and some number $x_1 > 0$ such that for all $x \geq x_1$,

$$\Pr\left(\Delta_{m_0} > x\right) \leq \varepsilon \Pr\left(X_1 \mathrm{e}^{-\delta\tau_1} > x\right), \tag{4.6}$$

and

$$\sum_{n=m_0+1}^{\infty} \Pr\left(X_n \mathrm{e}^{-\delta\tau_n} > x\right) \leq \sum_{n=m_0+1}^{\infty} \Pr\left(X_n \mathrm{e}^{-\delta\tau_n} > xn^{-2}\right) \leq \varepsilon \Pr\left(X_1 \mathrm{e}^{-\delta\tau_1} > x\right). \tag{4.7}$$

Let m_0 be fixed as above. Applying Lemma 2 we obtain that relation (3.5) holds. By (3.5) and (4.7),

$$\begin{aligned}
\Pr\left(\sum_{n=1}^{\infty} X_n \mathrm{e}^{-\delta\tau_n} > x\right) &\geq \Pr\left(\sum_{n=1}^{m_0} X_n \mathrm{e}^{-\delta\tau_n} > x\right) \\
&\sim \left(\sum_{n=1}^{\infty} - \sum_{n=m_0+1}^{\infty}\right) \Pr\left(X_n \mathrm{e}^{-\delta\tau_n} > x\right) \\
&\geq (1-\varepsilon) \sum_{n=1}^{\infty} \Pr\left(X_n \mathrm{e}^{-\delta\tau_n} > x\right).
\end{aligned} \tag{4.8}$$

For any $0 < \theta < 1/2$, by (3.5) and (4.6),

$$\begin{aligned}
\Pr\left(\sum_{n=1}^{\infty} X_n \mathrm{e}^{-\delta\tau_n} > x\right) &\leq \Pr\left(\sum_{n=1}^{m_0} X_n \mathrm{e}^{-\delta\tau_n} > (1-\theta)x\right) + \Pr\left(\sum_{n=m_0+1}^{\infty} X_n \mathrm{e}^{-\delta\tau_n} > \theta x\right) \\
&\lesssim \sum_{n=1}^{m_0} \Pr\left(X_n \mathrm{e}^{-\delta\tau_n} > (1-\theta)x\right) + \varepsilon \Pr\left(X_1 \mathrm{e}^{-\delta\tau_1} > \theta x\right) \\
&= J_1(m_0,\theta,x) + J_2(\theta,x).
\end{aligned} \tag{4.9}$$

Since Lemma 1 tells us that for every $n = 1, 2, \dots$ the distribution of the product $X_n \mathrm{e}^{-\delta\tau_n}$ still belongs to the class ERV $(-\alpha, -\beta)$, by (2.3) we have

$$J_1(m_0, \theta, x) \lesssim (1-\theta)^{-\beta} \sum_{n=1}^{m_0} \Pr\left(X_n \mathrm{e}^{-\delta\tau_n} > x\right).$$

Similarly,

$$J_2(\theta, x) \lesssim \varepsilon \theta^{-\beta} \Pr\left(X_1 \mathrm{e}^{-\delta\tau_1} > x\right).$$

From (4.9) we obtain that

$$\Pr\left(\sum_{n=1}^{\infty} X_n \mathrm{e}^{-\delta\tau_n} > x\right) \lesssim \left((1-\theta)^{-\beta} + \varepsilon\theta^{-\beta}\right) \sum_{n=1}^{\infty} \Pr\left(X_n \mathrm{e}^{-\delta\tau_n} > x\right). \tag{4.10}$$

Using the bounds given in (4.8) and (4.10) and taking into account the arbitrariness of ε and θ, we obtain that

$$\Pr\left(\sum_{n=1}^{\infty} X_n \mathrm{e}^{-\delta\tau_n} > x\right) \sim \sum_{n=1}^{\infty} \Pr\left(X_n \mathrm{e}^{-\delta\tau_n} > x\right). \tag{4.11}$$

Finally, calculating the probabilities $\Pr\left(X_n \mathrm{e}^{-\delta\tau_n} > x\right)$ by conditioning on τ_n, $n = 1, 2, \dots$, and recalling (1.1), we find that the right-hand side of (4.11) is equal to $\int_0^\infty \overline{F}(x\mathrm{e}^{\delta t})\mathrm{d}m_t$. This proves relation (4.2), as announced. □

Acknowledgments: We would like to thank the referee for his/her valuable comments by which we have improved the presentation of the paper.

References

[1] Asmussen, S. Subexponential asymptotics for stochastic processes: extremal behavior, stationary distributions and first passage probabilities. Ann. Appl. Probab. 8 (1998), no. 2, 354–374.

[2] Asmussen, S.; Bladt, M. Renewal theory and queueing algorithms for matrix-exponential distributions. Matrix-analytic methods in stochastic models (Flint, MI), 313–341, Lecture Notes in Pure and Appl. Math., 183, Dekker, New York, 1997.

[3] Bingham, N. H.; Goldie, C. M.; Teugels, J. L. Regular variation. Cambridge University Press, Cambridge, 1987.

[4] Cline, D. B. H.; Samorodnitsky, G. Subexponentiality of the product of independent random variables. Stochastic Process. Appl. 49 (1994), no. 1, 75–98.

[5] Kaas, R.; Tang, Q. A large deviation result for aggregate claims with dependent claim occurrences. Insurance Math. Econom. 36 (2005), no. 3, 251–259.

[6] Kalashnikov, V.; Konstantinides, D. Ruin under interest force and subexponential claims: a simple treatment. Insurance Math. Econom. 27 (2000), no. 1, 145–149.

[7] Klüppelberg, C.; Stadtmüller, U. Ruin probabilities in the presence of heavy-tails and interest rates. Scand. Actuar. J. (1998), no. 1, 49–58.

[8] Konstantinides, D.; Tang, Q.; Tsitsiashvili, G. Estimates for the ruin probability in the classical risk model with constant interest force in the presence of heavy tails. Insurance Math. Econom. 31 (2002), no. 3, 447–460.

[9] Kotz, S.; Balakrishnan, N.; Johnson, N. L. Continuous Multivariate Distributions. Vol. 1. Models and applications. Second edition. Wiley-Interscience, New York, 2000.

[10] Lehmann, E. L. Some concepts of dependence. Ann. Math. Statist. 37 (1966), no. 5, 1137–1153.

[11] Resnick, S. I.; Willekens, E. Moving averages with random coefficients and random coefficient autoregressive models. Communications in Statistics. Stochastic Models 7 (1991), no. 4, 511–525.

[12] Sundt, B.; Teugels, J. L. Ruin estimates under interest force. Insurance Math. Econom. 16 (1995), no. 1, 7–22.

[13] Tang, Q. Asymptotic ruin probabilities of the renewal model with constant interest force and regular variation. Scand. Actuar. J. (2005a), no. 1, 1–5.

[14] Tang, Q. The finite-time ruin probability of the compound Poisson model with constant interest force. J. Appl. Probab. 42 (2005b), no. 3, 608–619.

[15] Tang, Q.; Tsitsiashvili, G. Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks. Stochastic Process. Appl. 108 (2003), no. 2, 299–325.

[16] Tang, Q.; Tsitsiashvili, G. Finite- and infinite-time ruin probabilities in the presence of stochastic returns on investments. Adv. in Appl. Probab. 36 (2004), no. 4, 1278–1299.