On the Maximum of Randomly Weighted Sums with Regularly Varying Tails

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Abstract

Consider the randomly weighted sums $S_n(\theta) = \sum_{k=1}^{n} \theta_k X_k$, $n = 1, 2, \ldots$, where $\{X_k, k = 1, 2, \ldots\}$ is a sequence of independent real-valued random variables with common distribution F, whose right tail is regularly varying with exponent $-\alpha < 0$, and $\{\theta_k, k = 1, 2, \ldots\}$ is a sequence of positive random variables, independent of $\{X_k, k = 1, 2, \ldots\}$. Under a suitable summability condition on the upper endpoints of $\theta_k, k = 1, 2, \ldots$, we prove that $\Pr\left(\max_{1\le n<\infty} S_n(\theta) > x\right) \sim \overline{F}\,(x) \sum_{k=1}^{\infty} \mathrm{E}\theta_k^{\alpha}$.

Keywords: Asymptotics; Regular variation; Ruin probability; Tail probability.

Following some recent works in extreme value theory and applied probability, in this paper we are interested in the tail probability of the maximum of randomly weighted sums

$$S_n(\theta) = \sum_{k=1}^{n} \theta_k X_k, \qquad n = 1, 2, \ldots, \tag{1}$$

where $\{X_k, k = 1, 2, \ldots\}$ is a sequence of independent, identically distributed (i.i.d.), and real-valued random variables (r.v.'s) with generic r.v. X and common distribution function (d.f.) $F = 1 - \overline{F}$, and $\{\theta_k, k = 1, 2, \ldots\}$ is another sequence of positive r.v.'s. We assume that the two sequences are mutually independent, that each weight θ_k has an upper endpoint

$$c_k = c(\theta_k) = \sup\{c : \Pr(Y \le c) < 1\} < \infty, \qquad k = 1, 2, \ldots,$$

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and that the right tail of F is regularly varying in the sense that there exist some constant $0 < \alpha < \infty$ and some slowly varying function $L(\cdot)$ such that

$$\overline{F}(x) = x^{-\alpha} L(x), \qquad x > 0. \tag{2}$$

For simplicity we designate the fact in (2) by $F \in \mathcal{R}_{-\alpha}$. For detail of regular variation, the reader is referred to Bingham *et al.* (1987) and Embrechts *et al.* (1997).

Throughout, all limit relationships are for $x \to \infty$ unless otherwise stated. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) \lesssim b(x)$ if $\limsup a(x)/b(x) \le 1$, write $a(x) \gtrsim b(x)$ if $\liminf a(x)/b(x) \ge 1$, and write $a(x) \sim b(x)$ if both.

The main contribution of this paper is the following:

Theorem 1. *Consider the randomly weighted sums (1). If $F \in \mathcal{R}_{-\alpha}$ for some $0 < \alpha < \infty$ and*

$$\sum_{k=1}^{\infty} c_k^p < \infty \qquad \text{for some } 0 < p < \frac{\alpha}{1+\alpha}, \tag{3}$$

then

$$\Pr\left(\max_{1 \le n < \infty} \sum_{k=1}^{n} \theta_k X_k > x \right) \sim \overline{F}\,(x) \sum_{k=1}^{\infty} \mathrm{E}\theta_k^{\alpha}. \tag{4}$$

Note that in Theorem 1, we do not ask for any information about the dependence structure of the sequence $\{\theta_k, k = 1, 2, \ldots\}$. Related works can be found in Resnick and Willekens (1991), Tang and Tsitsiashvili (2003), Chen *et al.* (2005), among others.

We may give an actuarial explanation for Theorem 1, as done in Tang and Tsitsiashvili (2003). Consider a discrete-time risk model with stochastic interest rates. In this model the surplus process is expressed by a recursive equation

$$U_0 = x, \qquad U_n = U_{n-1}(1 + R_n) - X_n, \qquad n = 1, 2, \ldots, \tag{5}$$

where $x \ge 0$ denotes the initial surplus, $\{R_n, n = 1, 2, \ldots\}$ denotes a sequence of stochastic and positive interest rates, X_n denotes the gross loss (that is, the total claim amount minus the total incoming premium) during the nth time period, and X_n, $n = 1, 2, \ldots$, constitute a sequence of i.i.d. real-valued r.v.'s with common d.f. F. We assume that the sequences $\{X_n, n = 1, 2, \ldots\}$ and $\{R_n, n = 1, 2, \ldots\}$ are mutually independent. The ultimate ruin probability is defined by

$$\psi(x) = \Pr\left(\min_{0 \le n < \infty} U_n < 0 \mid U_0 = x \right).$$

Iterating (5) yields that

$$U_0 = x, \qquad U_n = x\prod_{k=1}^{n}(1+R_k) - \sum_{k=1}^{n} X_k \prod_{i=k+1}^{n}(1+R_i), \qquad n = 1, 2, \ldots.$$

Hence, the ruin probability $\psi(x)$ can be rewritten as

$$\psi(x) = \Pr\left(\max_{1\le n<\infty}\sum_{k=1}^{n} X_k \prod_{i=1}^{k}(1+R_i)^{-1} > x\right) = \Pr\left(\max_{1\le n<\infty}\sum_{k=1}^{n}\theta_k X_k > x\right),$$

where $\theta_k = \prod_{i=1}^{k}(1+R_i)^{-1}$ for $k = 1, 2, \ldots$. In this way, Theorem 1 gives an asymptotic formula for the ruin probability.

During the proof of Theorem 1, we shall need the following two lemmas:

Lemma 1. *Let $\{X_k, k = 1, 2, \ldots, n\}$ be n i.i.d. r.v.'s with common d.f. $F \in \mathcal{R}_{-\alpha}$ for some $0 < \alpha < \infty$, and let $\{d_k, k = 1, 2, \ldots, n\}$ be n positive constants. Then*

$$\Pr\left(\sum_{k=1}^{n} d_k X_k > x\right) \sim \overline{F}(x)\sum_{k=1}^{n} d_k^{\alpha}. \tag{6}$$

Proof. Since $F \in \mathcal{R}_{-\alpha}$, it holds for each $k = 1, 2, \ldots, n$ that

$$\Pr\left(d_k X_k > x\right) \sim d_k^{\alpha}\overline{F}(x).$$

Applying Lemma A3.28 in Embrechts *et al.* (1997), we obtain (6). □

The following lemma is a direct consequence of Potter's Theorem; see Bingham *et al.* (1987, Theorem 1.5.6(iii)):

Lemma 2. *If $F \in \mathcal{R}_{-\alpha}$ for some $0 < \alpha < \infty$, then for any constants $A > 1$ and $0 < v < \alpha$, there exists some $M = M(A, v) > 0$ such that the inequality*

$$\frac{\overline{F}(tx)}{\overline{F}(x)} \le At^{-v}$$

holds for all $tx \ge x \ge M$.

Proof of Theorem 1:

Denote by $x^+ = \max\{x, 0\}$ the positive part of a real number x. Trivially,

$$\Pr\left(\max_{1\le n<\infty}\sum_{k=1}^{n}\theta_k X_k > x\right) \lesssim \Pr\left(\sum_{k=1}^{\infty}\theta_k X_k^+ > x\right)$$

Thus, it suffices to prove

$$\Pr\left(\sum_{k=1}^{\infty}\theta_k X_k^+ > x\right) \lesssim \overline{F}(x)\sum_{k=1}^{\infty}\mathrm{E}\theta_k^{\alpha} \tag{7}$$

and

$$\Pr\left(\max_{1\le n<\infty}\sum_{k=1}^{n}\theta_k X_k > x\right) \gtrsim \overline{F}(x)\sum_{k=1}^{\infty}\mathrm{E}\theta_k^{\alpha}. \tag{8}$$

First we prove relation (7). For any $m = 1, 2, \ldots$ and any small $0 < l < 1$, we derive

$$\begin{aligned}\Pr\left(\sum_{k=1}^{\infty}\theta_k X_k^+ > x\right) &\le \Pr\left(\sum_{k=1}^{m}\theta_k X_k^+ > (1-l)x\right) + \Pr\left(\sum_{k=m+1}^{\infty}\theta_k X_k^+ > lx\right)\\ &= A_m + B_m.\end{aligned}$$

We show that B_m is asymptotically negligible in comparison to A_m when x and m are sufficiently large. Applying (3), we see that, for any $0 < \varepsilon < 1$, there is some integer $m = m(\varepsilon, l)$ sufficiently large such that the inequalities

$$\sum_{k=m+1}^{\infty} l^{-\alpha} c_k^{\alpha(1+\alpha)^{-1}} \le \varepsilon \qquad \text{and} \qquad l c_k^{-(1+\alpha)^{-1}} \ge 1 \quad \text{for all } k > m$$

hold simultaneously. With this fixed m we have

$$B_m \le \Pr\left(\sum_{k=m+1}^{\infty} c_k X_k^+ > \sum_{k=m+1}^{\infty} c_k^{\alpha(1+\alpha)^{-1}} lx\right) \le \sum_{k=m+1}^{\infty}\Pr\left(X^+ > l c_k^{-(1+\alpha)^{-1}} x\right).$$

Applying Lemma 2 with $v = p(1+\alpha) < \alpha$ and $t = l c_k^{-(1+\alpha)^{-1}}$, we see that the inequality

$$\frac{\Pr\left(X_k^+ > l c_k^{-(1+\alpha)^{-1}} x\right)}{\overline{F}(x)} \le A\left(l c_k^{-(1+\alpha)^{-1}}\right)^{-v} = A l^{-v} c_k^{p}$$

holds for some $A > 0$, all $k > m$, and all large x. Since $\sum_{k=1}^{\infty} A l^{-v} c_k^{p} < \infty$, applying Fatou's lemma we obtain that

$$\limsup_{x\to\infty}\frac{B_m}{\overline{F}(x)} \le \sum_{k=m+1}^{\infty}\limsup_{x\to\infty}\frac{\Pr\left(X^+ > l c_k^{-(1+\alpha)^{-1}} x\right)}{\overline{F}(x)} = \sum_{k=m+1}^{\infty}\left(l c_k^{-(1+\alpha)^{-1}}\right)^{-\alpha} \le \varepsilon. \tag{9}$$

We turn to derive an asymptotic relation for A_m. By Lemma 1 it is clear that

$$\Pr\left(\sum_{k=1}^{m}\theta_k X_k^+ > x \,\middle|\, \theta_1, \cdots, \theta_m\right) \le \Pr\left(\sum_{k=1}^{m} c_k X_k^+ > x\right) \sim \overline{F}(x)\sum_{k=1}^{m} c_k^{\alpha}. \tag{10}$$

Applying the dominated convergence theorem, the applicability of which is guaranteed by (10), and recalling Lemma 1, we obtain

$$\begin{aligned}\lim_{x\to\infty}\frac{A_m}{\overline{F}((1-l)x)} &= \lim_{x\to\infty}\frac{\mathrm{E}\left[\Pr\left(\sum_{k=1}^{m}\theta_k X_k^+ > (1-l)x \,\middle|\, \theta_1,\cdots,\theta_m\right)\right]}{\overline{F}((1-l)x)} \\ &= \mathrm{E}\left[\lim_{x\to\infty}\frac{\Pr\left(\sum_{k=1}^{m}\theta_k X_k^+ > (1-l)x \,\middle|\, \theta_1,\cdots,\theta_m\right)}{\overline{F}((1-l)x)}\right] \\ &= \mathrm{E}\left[\sum_{k=1}^{m}\theta_k^\alpha\right] = \sum_{k=1}^{m}\mathrm{E}\theta_k^\alpha. \qquad (11)\end{aligned}$$

It follows from (9) and (11) that

$$\begin{aligned}\Pr\left(\sum_{k=1}^{\infty}\theta_k X_k^+ > x\right) &\le A_m + B_m \lesssim \overline{F}((1-l)x)\sum_{k=1}^{\infty}\mathrm{E}\theta_k^\alpha + \varepsilon\overline{F}(x) \\ &\sim \overline{F}(x)\left((1-l)^{-\alpha}\sum_{k=1}^{\infty}\mathrm{E}\theta_k^\alpha + \varepsilon\right).\end{aligned}$$

The relation (7) follows from the arbitrariness of $0<\varepsilon, l<1$.

Next we prove (8). Let $n\ge 1$ be arbitrarily fixed. Applying Fatou's lemma and recalling Lemma 1 once again, we obtain

$$\begin{aligned}\liminf_{x\to\infty}\frac{\Pr\left(\max_{1\le n<\infty}\sum_{k=1}^{n}\theta_k X_k > x\right)}{\overline{F}(x)} &\ge \liminf_{x\to\infty}\frac{\mathrm{E}\left[\Pr\left(\sum_{k=1}^{n}\theta_k X_k > x \,\middle|\, \theta_1,\cdots,\theta_n\right)\right]}{\overline{F}(x)} \\ &\ge \mathrm{E}\left[\liminf_{x\to\infty}\frac{\Pr\left(\sum_{k=1}^{n}\theta_k X_k > x \,\middle|\, \theta_1,\cdots,\theta_n\right)}{\overline{F}(x)}\right] \\ &= \mathrm{E}\left[\sum_{k=1}^{n}\theta_k^\alpha\right] = \sum_{k=1}^{n}\mathrm{E}\theta_k^\alpha.\end{aligned}$$

Since the integer $n\ge 1$ are arbitrary and the series $\sum_{k=1}^{\infty}\mathrm{E}\theta_k^\alpha$ converges, we obtain (8). □

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