Explicit Asymptotics for the Ruin Probability with Risky Investment Included*

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Abstract

In this paper, we investigate the ruin probability of a discrete-time risk model, in which the surplus of an insurance business is currently invested into a risky asset. Using a purely probabilistic treatment, we establish explicit asymptotic relations for the infinite-time ruin probabilities, hence we extend a recent result of Tang and Tsitsiashvili (2003) to the infinite-time case.

Keywords: Asymptotics; Regular variation; Ruin probability; Stochastic equation.

1 Introduction

Following Nyrhinen (1999), Tang and Tsitsiashvili (2003), and Chen and Xie (2005), we consider a discrete-time risk model, in which the surplus of the insurance company is currently invested into a risky asset which may lead to a negative return in each year. Denote by $A_n \in (-\infty, \infty)$ the net income (the total incoming premium minus the total claim amount) within year n and by $r_n \in (-1, \infty)$ the return rate at year n, $n = 1, 2, \ldots$. Let the initial surplus be $x \geq 0$. Hence, if we assume that the net income A_n is calculated at the beginning of year n, then the surplus accumulated till the end of year n, characterized by S_n, satisfies the recurrence equation

$$S_0 = x \geq 0, \quad S_n = (1 + r_n)(S_{n-1} + A_n), \qquad n = 1, 2, \ldots; \tag{1.1}$$

alternatively, if we assume that the net income A_n is calculated at the end of year n, then the surplus accumulated till the end of year n, characterized by T_n, satisfies the recurrence equation

$$T_0 = x \geq 0, \quad T_n = (1 + r_n)T_{n-1} + A_n, \qquad n = 1, 2, \ldots. \tag{1.2}$$

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Throughout the paper, we assume that the net incomes A_n, $n = 1, 2, \ldots$, constitute a sequence of independent, identically distributed (i.i.d.) random variables (r.v.'s), that the return rates r_n, $n = 1, 2, \ldots$, also constitute a sequence of i.i.d. r.v.'s, and that the two sequences $\{A_n, n = 1, 2, \ldots\}$ and $\{r_n, n = 1, 2, \ldots\}$ are independent.

Write

$$X_n = -A_n, \quad Y_n = \frac{1}{1 + r_n}, \qquad n = 1, 2, \ldots. \tag{1.3}$$

The r.v. X_n is the net payout during year n and the r.v. Y_n is the discount factor from year n to year $n-1$, $n = 1, 2, \ldots$. In what follows, we write by X and Y the generic r.v.'s of the sequences $\{X_n, n = 1, 2, \ldots\}$ and $\{Y_n, n = 1, 2, \ldots\}$ and write by F and G the distribution functions (d.f.'s) of the r.v.'s X and Y, respectively. Clearly, the r.v. Y is strictly positive. We assume that the tail probability $\overline{F}(x) = 1 - F(x) = \Pr(X > x)$ satisfies $\overline{F}(x) > 0$ for any real number x. In the terminology of Norberg (1999) and Tang and Tsitsiashvili (2003), we call X the insurance risk and Y the financial risk.

Corresponding to the surplus processes (1.1) and (1.2), we define the ruin probabilities within finite time $n = 1, 2, \ldots$ and infinite time as

$$\psi_S(x, n) = \Pr\left(\min_{0 \le k \le n} S_k < 0 \mid S_0 = x \right), \quad \psi_T(x, n) = \Pr\left(\min_{0 \le k \le n} T_k < 0 \mid T_0 = x \right), \tag{1.4}$$

respectively,

$$\psi_S(x) = \Pr\left(\min_{0 \le k < \infty} S_k < 0 \mid S_0 = x \right), \quad \psi_T(x) = \Pr\left(\min_{0 \le k < \infty} T_k < 0 \mid T_0 = x \right). \tag{1.5}$$

In this paper we are interested in the asymptotic behavior of these probabilities.

Hereafter, all limit relationships are for $x \to \infty$ unless stated otherwise; for two positive functions $a(x)$ and $b(x)$, we write $a(x) \lesssim b(x)$ if $\limsup a(x)/b(x) \le 1$, write $a(x) \gtrsim b(x)$ if $\liminf a(x)/b(x) \ge 1$, and write $a(x) \sim b(x)$ if both.

Nyrhinen (1999) investigated the asymptotic behavior of the ruin probability $\psi_S(x)$. In terms of the model described above, we can obtain a combination of Theorems 3.3 and 3.4 of Nyrhinen (1999) as follows: if (1) $w = \sup\{t \mid \mathrm{E}Y^t \le 1\} \in (0, \infty)$, (2) $\mathrm{E}Y^t$ and $\mathrm{E}\,|X|^t$ are finite for some $t > w$, (3) $\overline{F}(0) > 0$, and (4) some of the convolution powers of the distribution of $\log Y$ has a non-trivial absolutely continuous component, then the relation

$$\psi_S(x) \sim C x^{-w} \tag{1.6}$$

holds for some positive, but implicit, constant C.

The objective of the present paper is to establish explicit asymptotic relations for the ruin probabilities $\psi_S(x)$ and $\psi_T(x)$ under some other assumptions on the tails of the r.v.'s X and Y. In the proof, we will use a recent result of Tang and Tsitsiashvili (2003), who investigated the finite-time ruin probabilities $\psi_S(x, n)$ and $\psi_T(x, n)$ under the assumptions that F is heavy tailed and that the tail $\overline{G}$ is dominated by the tail $\overline{F}$. We will also apply some results obtained by Vervaat (1979) in the literature of stochastic difference equations.

The rest of this paper consists of three sections: Section 2 presents the main results after recalling an important class of heavy-tailed distributions, Section 3 collects some lemmas, and Section 4 proves the theorems.

2 Main results

We will assume that the d.f. F of the insurance risk X has a regularly varying tail, denoted by $F \in \mathcal{R}$. By definition, a d.f. F concentrated on $(-\infty, \infty)$ belongs to the class $\mathcal{R}$ if there is some $\alpha \geq 0$ such that

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha} \tag{2.1}$$

for any $y > 0$. For simplicity, we denote by $F \in \mathcal{R}_{-\alpha}$ the regularity property in (2.1). In this case we have

$$\overline{F}(x) = x^{-\alpha} c(x) \exp\left\{ \int_a^x \frac{\varepsilon(y)}{y} \mathrm{d}y \right\}, \qquad x > a, \tag{2.2}$$

for some $a > 0$, where $c(x) \to c \in (0, \infty)$ and $\varepsilon(x) \to 0$; see Bingham *et al.* (1987, page 21). The corresponding r.v. X satisfies

$$\mathrm{E}\left(X^+\right)^p < \infty \;\text{ for } 0 \leq p < \alpha, \qquad \mathrm{E}\left(X^+\right)^p = \infty \;\text{ for } p > \alpha, \tag{2.3}$$

where $x^+ = \max\{x, 0\}$.

Under the assumptions that F belongs to a heavy-tailed distribution class, which is slightly larger than the class $\mathcal{R}$, and that the tail $\overline{G}$ is dominated by the tail $\overline{F}$ (to be precise, $\mathrm{E}Y^p < \infty$ for some p larger than the upper Matuszewska index of the d.f. F), Tang and Tsitsiashvili (2003, Theorem 5.1 and Remark 5.1) obtained precise asymptotic estimates for the finite-time ruin probabilities $\psi_S(x, n)$ and $\psi_T(x, n)$ for each fixed $n = 1, 2, \ldots$. The really applicable results of their paper were obtained in the case that

$$F \in \mathcal{R}_{-\alpha} \quad \text{and} \quad \mathrm{E}Y^p < \infty \qquad \text{for some } p > \alpha > 0. \tag{2.4}$$

In this case, it holds for each fixed $n = 1, 2, \ldots$ that

$$\psi_S(x, n) \sim \frac{1 - (\mathrm{E}Y^\alpha)^n}{1 - \mathrm{E}Y^\alpha} \overline{F}(x); \tag{2.5}$$

see Tang and Tsitsiashvili (2003, Theorem 5.2(3) and Remark 5.1).

If further we assume that

$$\mathrm{E}Y^\alpha < 1, \tag{2.6}$$

then the term $(\mathrm{E}Y^{\alpha})^n$ in relation (2.5) vanishes as n increases. Though relation (2.5) was only proved by Tang and Tsitsiashvili (2003) to hold for fixed n, we intuitively believe that the infinite-time ruin probability $\psi_S(x)$ should satisfy

$$\psi_S(x) \sim \frac{1}{1-\mathrm{E}Y^{\alpha}}\overline{F}(x). \tag{2.7}$$

The following result proves that this is true.

Theorem 2.1. *Under assumptions (2.4) and (2.6), the asymptotic relation (2.7) holds.*

Theorem 2.1 successfully establishes an asymptotic relation for the ruin probability $\psi_S(x)$ in a fully explicit form.

As for the ruin probability $\psi_T(x)$, we have a similar explicit asymptotic relation below.

Theorem 2.2. *Under assumptions (2.4) and (2.6), it holds that*

$$\psi_T(x) \sim \frac{\mathrm{E}Y^{\alpha}}{1-\mathrm{E}Y^{\alpha}}\overline{F}(x). \tag{2.8}$$

3 Some lemmas

We first point out a simple relationship between the ruin probabilities indexed by S and those indexed by T.

Lemma 3.1. *For the risk model introduced in Section 1, the ruin probabilities defined by (1.4) and (1.5) satisfy the relations*

$$\psi_T(x,n) = \int_0^{\infty} \psi_S\left(x/y,n\right) G(\mathrm{d}y), \qquad n=1,2,\ldots, \tag{3.1}$$

and

$$\psi_T(x) = \int_0^{\infty} \psi_S\left(x/y\right) G(\mathrm{d}y). \tag{3.2}$$

Proof. Iterating the recurrence equation (1.1) yields that

$$S_0 = x, \quad S_n = x\prod_{j=1}^{n}(1+r_j) + \sum_{i=1}^{n} A_i \prod_{j=i}^{n}(1+r_j), \qquad n=1,2,\ldots. \tag{3.3}$$

We write the discounted value of the surplus S_n in (3.3) as

$$\widetilde{S}_0 = x, \qquad \widetilde{S}_n = S_n\prod_{j=1}^{n} Y_j = x + \sum_{i=1}^{n} A_i \prod_{j=1}^{i-1} Y_j = x - \sum_{i=1}^{n} X_i \prod_{j=1}^{i-1} Y_j,$$

where $\prod_{j=1}^{0} = 1$ by convention. It is clear that for each $n=1,2,\ldots,$

$$\psi_S(x,n) = \Pr\left(\max_{1\le k\le n}\sum_{i=1}^{k} X_i \prod_{j=1}^{i-1} Y_j > x\right), \quad \psi_S(x) = \Pr\left(\max_{1\le k<\infty}\sum_{i=1}^{k} X_i \prod_{j=1}^{i-1} Y_j > x\right). \tag{3.4}$$

Similarly, it holds that

$$\psi_T(x,n) = \Pr\left(\max_{1\le k\le n}\sum_{i=1}^{k} X_i \prod_{j=1}^{i} Y_j > x\right), \quad \psi_T(x) = \Pr\left(\max_{1\le k<\infty}\sum_{i=1}^{k} X_i \prod_{j=1}^{i} Y_j > x\right). \tag{3.5}$$

Hence by the i.i.d. assumptions made in Section 1, relations (3.1) and (3.2) holds. □

The lemma below is a combination of several results of Vervaat (1979).

Lemma 3.2. *Let* $\{(\widetilde{X}_n, \widetilde{Y}_n), n = 1, 2, \ldots\}$ *be a sequence of i.i.d. random pairs with generic random pair* $(\widetilde{X}, \widetilde{Y})$*. Consider the stochastic difference equation*

$$\widetilde{V}_n = \widetilde{Y}_n \widetilde{V}_{n-1} + \widetilde{X}_n, \qquad n = 1, 2, \ldots. \tag{3.6}$$

If $-\infty \le \mathrm{E}\log \widetilde{Y} < 0$ *and* $\mathrm{E}\left(\log\left|\widetilde{X}\right|\right)^{+} < \infty$*, then the r.v.'s* $\widetilde{V}_n$ *converges in distribution to some real-valued r.v.* V_∞*, which is invariant in distribution for all initial r.v.'s* $\widetilde{V}_0$*.*

Proof. By Theorem 1.6(b,c) of Vervaat (1979), the stochastic equation

$$\widetilde{V} =^d \widetilde{Y}\widetilde{V} + \widetilde{X}, \qquad \widetilde{V} \text{ is independent of } (\widetilde{X}, \widetilde{Y}), \tag{3.7}$$

has a solution, where $=^d$ denotes equality in distribution. Then, by Theorem 1.5(i) of Vervaat (1979), this solution is unique in distribution and $\widetilde{V}_n$ defined by (3.6) converges in distribution to some $V_\infty(\widetilde{V}_0)$, say, for any initial r.v. $\widetilde{V}_0$. Since, by Lemma 1.1 of Vervaat (1979), any limit r.v. $V_\infty(\widetilde{V}_0)$ should be a solution of equation (3.7), we prove that for all $\widetilde{V}_0$ the r.v.'s $V_\infty(\widetilde{V}_0)$ are identical in distribution. □

The following lemma is well known and is from Proposition of Feller (1971, p. 278) or Lemma 1.3.1 of Embrechts *et al.* (1997).

Lemma 3.3. *Let* F_1 *and* F_2 *be two d.f.'s concentrated on* $[0, \infty)$*. If* $F_1 \in \mathcal{R}_{-\alpha}$ *and* $F_2 \in \mathcal{R}_{-\alpha}$ *for some* $\alpha \ge 0$*, then their convolution* $F_1 * F_2 \in \mathcal{R}_{-\alpha}$ *and*

$$\overline{F_1 * F_2}(x) \sim \overline{F}_1(x) + \overline{F}_2(x).$$

The following lemma is from Breiman (1965).

Lemma 3.4. *Let* X *and* Y *be two independent r.v.'s distributed by* F *and* G*, respectively, where* Y *is nonnegative. If d.f.* $F \in \mathcal{R}_{-\alpha}$ *for some* $0 < \alpha < \infty$ *and* $\mathrm{E}Y^p < \infty$ *for some* $p > \alpha$*, then*

$$\lim_{x\to\infty} \frac{\Pr(XY > x)}{\Pr(X > x)} = \mathrm{E}Y^\alpha.$$

4 Proofs of the main results

4.1 Proof of Theorem 2.1

From (3.4) and (2.5), it is clear that for each $n = 1, 2, \ldots$,

$$\psi_S(x) \geq \psi_S(x, n) \sim \frac{1 - (\mathrm{E}Y^\alpha)^n}{1 - \mathrm{E}Y^\alpha} \overline{F}(x).$$

It follows that for each $n = 1, 2, \ldots$,

$$\liminf_{x \to \infty} \frac{\psi_S(x)}{\overline{F}(x)} \geq \frac{1 - (\mathrm{E}Y^\alpha)^n}{1 - \mathrm{E}Y^\alpha}.$$

Letting $n \to \infty$ on the right-hand side,

$$\psi_S(x) \gtrsim \frac{1}{1 - \mathrm{E}Y^\alpha} \overline{F}(x). \tag{4.1}$$

It remains to derive a corresponding asymptotic upper bound for the ruin probability $\psi_S(x)$. To this end, from (3.4) we derive that

$$\psi_S(x) \leq \Pr\left(\sum_{i=1}^{\infty} X_i^+ \prod_{j=1}^{i-1} Y_j > x \right). \tag{4.2}$$

For each $n = 1, 2, \ldots, \infty$, set

$$U_n = \sum_{i=1}^{n} X_i^+ \prod_{j=1}^{i-1} Y_j.$$

By the i.i.d. assumptions, it is easy to see that for each $n = 1, 2, \ldots$,

$$U_n =^d \sum_{i=1}^{n} X_i^+ \prod_{j=i+1}^{n} Y_j = V_n, \tag{4.3}$$

where $\prod_{j=n+1}^{n} Y_j$ is equal to 1 by convention. Clearly, with $V_0 = 0$, the sequence $\{V_n, n = 1, 2, \ldots\}$ satisfies the recurrence equation

$$V_n = Y_n V_{n-1} + X_n^+ \qquad \text{for } n = 1, 2, \ldots. \tag{4.4}$$

Now we check the convergence in distribution of the sequence $\{V_n, n = 1, 2, \ldots\}$. By assumption (2.6) we easily understand that $-\infty \leq \mathrm{E}\log Y < 0$ should hold since the function $f(t) = \mathrm{E}Y^t$ is convex in $t \in [0, \alpha]$ and $f'_+(0) = \mathrm{E}\log Y$. Hence by Lemma 3.2, V_n converges in distribution to a r.v. V_∞, say, which is invariant for all V_0 In view of (4.3), this actually proves that U_∞ is finite almost surely and is equal to V_∞ in distribution. Specifically, we choose V_∞ to be independent of the sequences $\{X_n, n = 1, 2, \ldots\}$ and $\{Y_n, n = 1, 2, \ldots\}$.

Knowing $F \in \mathcal{R}_{-\alpha}$, we announce that

$$\Pr(U_\infty > x) = \Pr(V_\infty > x) \le \Pr(V_0 > x), \qquad x \ge 0, \tag{4.5}$$

for some nonnegative initial r.v. V_0 with a d.f. from the class $\mathcal{R}_{-\alpha}$. For this purpose, we choose a nonnegative r.v. Z independent of the sequences $\{X_n, n = 1, 2, \ldots\}$ and $\{Y_n, n = 1, 2, \ldots\}$ such that

$$\Pr(Z > x) \sim c\overline{F}(x)$$

for some positive constant c, which will be specified later. By Lemma 3.4 we know that the d.f. of the r.v. $Y_1 Z$ belongs to the class $\mathcal{R}_{-\alpha}$ and

$$\Pr(Y_1 Z > x) \sim c\mathrm{E}Y^\alpha \overline{F}(x);$$

by this and Lemma 3.3 we further know that

$$\Pr\left(Y_1 Z + X_1^+ > x\right) \sim (c\mathrm{E}Y^\alpha + 1)\,\overline{F}(x).$$

Hence, if we choose $c > 0$ sufficiently large such that $c\mathrm{E}Y^\alpha + 1 < c$, then there is some constant $x_0 > 0$ such that for all $x > x_0$,

$$\Pr\left(Y_1 Z + X_1^+ > x\right) \le \Pr(Z > x).$$

By this inequality we can prove that for all $x \ge 0$

$$\Pr\left(Y_1 Z + X_1^+ > x \middle| Z > x_0\right) \le \Pr(Z > x | Z > x_0). \tag{4.6}$$

In fact, for $0 \le x \le x_0$, inequality (4.6) trivially holds since $\Pr(Z > x | Z > x_0) = 1$; for $x > x_0$, inequality (4.6) can be verified in the following way:

$$\begin{aligned}
\Pr\left(Y_1 Z + X_1^+ > x \middle| Z > x_0\right) &= \frac{\Pr\left(Y_1 Z + X_1^+ > x, Z > x_0\right)}{\Pr(Z > x_0)} \\
&\le \frac{\Pr\left(Y_1 Z + X_1^+ > x\right)}{\Pr(Z > x_0)} \\
&\le \frac{\Pr(Z > x)}{\Pr(Z > x_0)} \\
&= \Pr(Z > x | Z > x_0).
\end{aligned}$$

We identify the initial r.v. V_0 such that it is independent of the sequences $\{X_n, n = 1, 2, \ldots\}$ and $\{Y_n, n = 1, 2, \ldots\}$ and is equal in distribution to the r.v. Z conditional on $(Z > x_0)$. Hence,

$$\Pr(V_0 > x) = \Pr(Z > x | Z > x_0) \sim \frac{c}{\Pr(Z > x_0)} \overline{F}(x). \tag{4.7}$$

It follows from (4.6) that for all $x \geq 0$,

$$\begin{aligned}\Pr(V_1 > x) &= \Pr\left(Y_1 V_0 + X_1^+ > x\right) \\ &= \Pr\left(Y_1 Z + X_1^+ > x \middle| Z > x_0\right) \\ &\leq \Pr(V_0 > x).\end{aligned}$$

Successively,

$$\begin{aligned}\Pr(V_2 > x) &= \Pr\left(Y_2 V_1 + X_2^+ > x\right) \\ &\leq \Pr\left(Y_2 V_0 + X_2^+ > x\right) \\ &= \Pr\left(Y_1 V_0 + X_1^+ > x\right) \\ &= \Pr(V_1 > x) \\ &\leq \Pr(V_0 > x).\end{aligned}$$

Applying the mathematical induction method we know that for all $n = 1, 2, \ldots$ and all $x \geq 0$,

$$\Pr(V_n > x) \leq \cdots \leq \Pr(V_2 > x) \leq \Pr(V_1 > x) \leq \Pr(V_0 > x).$$

Since V_n converges in distribution to V_∞, taking $n \to \infty$ yields that for all $x \geq 0$,

$$\Pr(U_\infty > x) = \Pr(V_\infty > x) \leq \Pr(V_0 > x),$$

as announced in (4.5).

We continue the proof of Theorem 2.1. For any $\varepsilon \in (0, 1)$ and any $n = 1, 2, \ldots$, we split the probability on the right-hand side of inequality (4.2) into two parts as

$$\begin{aligned}\psi_S(x) &\leq \Pr\left(\sum_{i=1}^{n} X_i^+ \prod_{j=1}^{i-1} Y_j > (1-\varepsilon)x\right) + \Pr\left(\sum_{i=n+1}^{\infty} X_i^+ \prod_{j=1}^{i-1} Y_j > \varepsilon x\right) \\ &= I_1(x, \varepsilon, n) + I_2(x, \varepsilon, n).\end{aligned}$$

Clearly, from (2.5) and (3.4) with X_i being replaced by X_i^+, we obtain that

$$I_1(x, \varepsilon, n) \sim \frac{1 - (\mathrm{E}Y^\alpha)^n}{1 - \mathrm{E}Y^\alpha} \overline{F}((1-\varepsilon)x).$$

Since V_∞ is independent of the sequences $\{X_n, n = 1, 2, \ldots\}$ and $\{Y_n, n = 1, 2, \ldots\}$,

$$\begin{aligned}I_2(x, \varepsilon, n) &= \Pr\left(\left(\sum_{i=n+1}^{\infty} X_i^+ \prod_{j=n+1}^{i-1} Y_j\right) \prod_{j=1}^{n} Y_j > \varepsilon x\right) \\ &= \Pr\left(V_\infty \prod_{j=1}^{n} Y_j > \varepsilon x\right) \\ &\leq \Pr\left(V_0 \prod_{j=1}^{n} Y_j > \varepsilon x\right) \\ &\sim \frac{c\,(\mathrm{E}Y^\alpha)^n}{\Pr(Z > x_0)} \overline{F}(\varepsilon x),\end{aligned}$$

where in the last step we applied relation (4.7) and Lemma 3.4 Thus, for any $\varepsilon \in (0,1)$ and any $n = 1, 2, \ldots$,

$$\psi_S(x) \lesssim \frac{1-(\mathrm{E}Y^\alpha)^n}{1-\mathrm{E}Y^\alpha}\overline{F}((1-\varepsilon)x) + \frac{c\,(\mathrm{E}Y^\alpha)^n}{\Pr(Z > x_0)}\overline{F}(\varepsilon x).$$

It follows from $F \in \mathcal{R}_{-\alpha}$ that

$$\begin{aligned}\limsup_{x\to\infty}\frac{\psi_S(x)}{\overline{F}(x)} &\le \frac{1-(\mathrm{E}Y^\alpha)^n}{1-\mathrm{E}Y^\alpha}\limsup_{x\to\infty}\frac{\overline{F}((1-\varepsilon)x)}{\overline{F}(x)} + \frac{c\,(\mathrm{E}Y^\alpha)^n}{\Pr(Z > x_0)}\limsup_{x\to\infty}\frac{\overline{F}(\varepsilon x)}{\overline{F}(x)} \\ &= \frac{1-(\mathrm{E}Y^\alpha)^n}{1-\mathrm{E}Y^\alpha}(1-\varepsilon)^{-\alpha} + \frac{c\,(\mathrm{E}Y^\alpha)^n}{\Pr(Z > x_0)}\varepsilon^{-\alpha}.\end{aligned}$$

Since n and ε are arbitrary and $\mathrm{E}Y^\alpha < 1$, first letting $n \to \infty$ and then letting $\varepsilon \to 0$ lead to the desired result that

$$\psi_S(x) \lesssim \frac{1}{1-\mathrm{E}Y^\alpha}\overline{F}(x). \tag{4.8}$$

Combining (4.1) and (4.8) we obtain (2.7). This ends the proof. □

4.2 Proof of Theorem 2.2

Introduce a survival d.f. R_S by $R_S(x) = (1-\psi_S(x))\,1_{(x\ge 0)}$, which is a standard d.f. concentrated on $[0,\infty)$ with a mass $R_S(\{0\}) = 1-\psi_S(0)$ at 0. Theorem 2.1 has proved that $R_S \in \mathcal{R}_{-\alpha}$. Hence, applying Lemma 3.4 to relation (3.2) yields that

$$\psi_T(x) \sim \mathrm{E}Y^\alpha\psi_S(x) \sim \frac{\mathrm{E}Y^\alpha}{1-\mathrm{E}Y^\alpha}\overline{F}(x).$$

This ends the proof. □

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